EXAMINATION AND EXPERIMENTAL EVALUATION OF A MATHEMATICAL ANALYSIS OF PULSE TUBE REFRIGERATION

by

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B.S.E.E., Virginia Polytechnic Institute

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June, 1966

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Department of Naval Architecture
and Marine Engineering
20 May, 1966

Certified by ______________________________

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Chairman, Departmental Committee
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OF A MATHEMATICAL ANALYSIS OF PULSE TUBE REFRIGERATION

by

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ABSTRACT

An examination of a mathematical analysis of Pulse Tube refrigeration as proposed by Rea was carried out. Rea originally investigated the performance of thermal regenerators subjected to rapid pressure and flow cycling and found that this class of regenerators is characterized by two basic parameters;

(1) The ratio of the convective heat transfer within the regenerator to the heat generated by compression,

(2) The ratio of the rate of mass storage within the void volume of the regenerator to the through-flow.

Rea's proposal was to treat the Pulse Tube as a very efficient regenerator in series with a very inefficient regenerator, and, thus, characterize the performance of the Pulse Tube by the same two parameters.

The examination of Rea's proposal indicates that, in addition to the two parameters which describe the performance of an efficient regenerator, there is, for the case of the inefficient regenerator, a third parameter which depends upon the average pressure and the rate at which the pressure and mass flow cycle.

An experimental evaluation of Rea's proposal modified to include the third parameter was performed on a Pulse Tube consisting of a spherical matrix regenerator in series with a void tube. There was good agreement observed between theoretical and experimental results.

Thesis Supervisor: Joseph L. Smith, Jr.

Title: Associate Professor of Mechanical Engineering

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The computer work associated with this project was done at the Computation Center of M.I.T.

NOMENCLATURE

A - cross-sectional area of the regenerator or pulse tube, ft^2.

A_o - cross-sectional area of the regenerator or pulse tube open to flow ($A_o = \in A$), ft^2.

A_T - heat transfer surface area per foot of length of regenerator or pulse tube, ft^2/ft.

B - dimensionless variable defined by equation (A.48).

C - dimensionless constant defined by equation (2.31).

C_o - integration constant ($C_o = C_1 \tau_c = C_1 \tau_e$).

C_1 - constant defined by equation (2.19), lb_m - $^oF/hr$.

c_m - specific heat of regenerator matrix or pulse tube wall, BTU/lb_m - ^oF.

c_p - constant pressure specific heat of the gas, BTU/lb_m - ^oF.

c_v - constant volume specific heat of the gas, BTU/lb_m - ^oF.

h - gas enthalpy at a position x along the regenerator or pulse tube, BTU/lb_m.

h, h_T - heat transfer coefficient at position x along the regenerator or pulse tube, $BTU/hr\text{-}ft^2$ - ^oF.

K - heat transfer coefficient, $BTU/hr\text{-}ft^2\text{-}^oF$.

k - thermal conductivity, $BTU/hr\text{-}ft\text{-}^oF$.

L - regenerator or pulse tube length, ft.

$\dot{m}$ - gas mass flow at position x along regenerator or pulse tube, lb_m/hr.

m - dimensionless mass flow ($m = \dot{m}/\dot{m}_o$).

$\dot{m}_o$ - mass flow at $x = 0$ in regenerator or pulse tube, lb_m/hr.

M - matrix mass per foot of regenerator length or mass per foot of pulse tube length, lb_m/ft.

n - positive exponent whose value depends on regenerator matrix ($n = 0$ for laminar flow pulse tube).

p - gas pressure in regenerator or pulse tube, lb_f/ft^2.

$\dot{Q}$ - heat flux, BTU/hr.

R - gas constant, $ft\text{-}lb_f/lb_m\text{-}^oF$.

t - time, hours.

T - temperature of the regenerator matrix or pulse tube wall at position x, ^oF.

T_g - temperature of the gas at position x along the regenerator or pulse tube, ^oF.

u - internal energy of the gas, BTU/lb_m.

V - volume, ft^3.

V_H - dead volume at end of pulse tube, ft^3.

V_{pt} - volume of pulse tube, ft^3.

V_R - void volume of regenerator, ft^3.

V_T - total volume of the regenerator or pulse tube, ft^3.

w - dimensionless distance along regenerator or pulse tube defined by equation (2.26).

x - distance along regenerator or pulse tube, ft.

Greek Letters

α - dimensionless Pulse Tube parameter defined by equation (3.18).

β - heat transfer area per unit total volume of the regenerator or pulse tube ($\beta = \frac{A_T L}{V}$), ft^2/ft^3.

ϵ - dimensionless regenerator or pulse tube void fraction.

γ - dimensionless ratio of specific heats ($\gamma = c_p/c_v$).

Φ - dimensionless pulse tube parameter defined by equation (2.35).

ρ - gas density, lb_m/ft^3.

Θ - dimensionless matrix or pulse tube wall temperature ($\Theta = T/T_o$).

τ - duration of cycle, hours.

τ_c - duration of a compression period, hours.

τ_e - duration of an expansion period, hours.

ψ - percentage of the total cycle time devoted to compression or expansion.

Subscripts

avg - average.

c - compression.

C - conditions at cold end of Pulse Tube.

cond - conduction.

e - expansion.

eff - effective.

exper - experimental.

g - gas.

heater - used with $\dot{Q}$ to indicate heater input to Pulse Tube.

H - conditions at hot end of pulse tube.

losses - used with $\dot{Q}$ to indicate conduction and radiation losses in Pulse Tube.

m - refers to regenerator matrix or pulse tube wall.

max - refers to maximum value of the variable w or p.

min - refers to minimum value of the variable p.

o - pulse tube or regenerator conditions at $x = w = 0$.

pt - pulse tube.

rad - radiation.

R - regenerator; also refers to refrigeration ($\mathring{Q}_R$).

theor - theoretical.

W - conditions at warm end of regenerator.

CHAPTER I

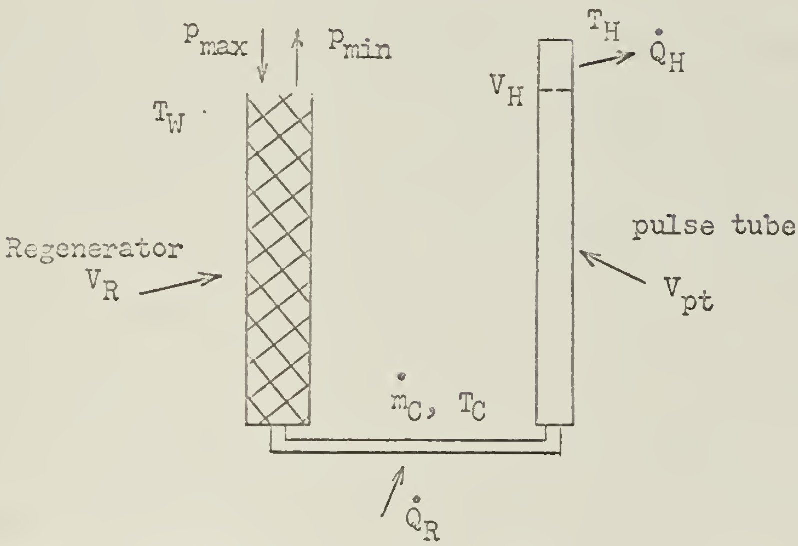
INTRODUCTION

General

A recent paper by Gifford and Longsworth (1)* describes an interesting thermodynamic effect which frequently is ignored in practice, but which has a potentially useful application. As Gifford and

FIGURE I

SCHEMATIC OF THE PULSE TUBE



Longsworth indicate (2), it is not generally recognized that the cycling of a gas in a closed volume can establish large temperature gradients within such a volume. Gifford and Longsworth, by suitably arranging a thermal regenerator, heat exchangers, and a void volume, have created a

* Numbers in ( ) refer to References.

device which preserves these temperature gradients in an essentially static state, even though the gas is undergoing rapid cycling. They have called this device a Pulse Tube. It is illustrated schematically in Figure I.

The construction and operation of the device are simple. This simplicity when coupled with the promising experimental results obtained thus far, indicates that the Pulse Tube should find practical application in the near future, perhaps as a small cooler in electronic equipment. Gifford and Longsworth (2) report refrigeration across temperature intervals as large as 209° F. with a single-stage system. The units may also be operated in multi-stages, so larger temperature differences are entirely feasible.

The Pulse Tube consists of a regenerator connected in series with a void tube, which is the pulse tube.** The void volume of the regenerator is V_R. The volume of the pulse tube is V_{pt}. Gas enters and exits the device through the warm end of the regenerator, which is at temperature T_W. The volume V_H, which is connected to the end of the pulse tube, is maintained at a constant temperature, T_H. This may be accomplished by circulating cooling water around the volume V_H.

The device operates in the following manner: the inlet valve opens admitting high pressure gas into the regenerator, thereby inducing a mass flow into the regenerator and pulse tube.

** Pulse Tube capitalized refers to the entire device (regenerator plus pulse tube); pulse tube without capital letters refers to the void tube. Wherever the void tube is intended, but grammatically capital letters are required, the following notation will be used: "pulse tube".

Concurrently, there is a continuous pressure rise throughout the Pulse Tube. At some predetermined maximum pressure, p_{max}, within the device, the inlet valve closes and the exit valve opens. This allows the gas within the Pulse Tube to flow back out through the warm end of the regenerator.

When the inlet valve is open, the gas in the pulse tube is heated as it is compressed by the entering gas. Some of this heated gas flows into the volume V_H, where it is cooled by the cooling medium to the temperature T_H. When the exit valve opens, the gas is cooled further by expansion as it flows back out of the volume V_H. Some of this cool gas must precool the entering warm gas during the next compression half-cycle. The remainder of the cool gas is available for refrigeration. Refrigeration becomes available, once a steady cyclic state is reached, at the connecting tubing between the regenerator and the pulse tube. This refrigeration load is labeled $\dot{Q}_R$ in Figure I and is available at the temperature T_C.

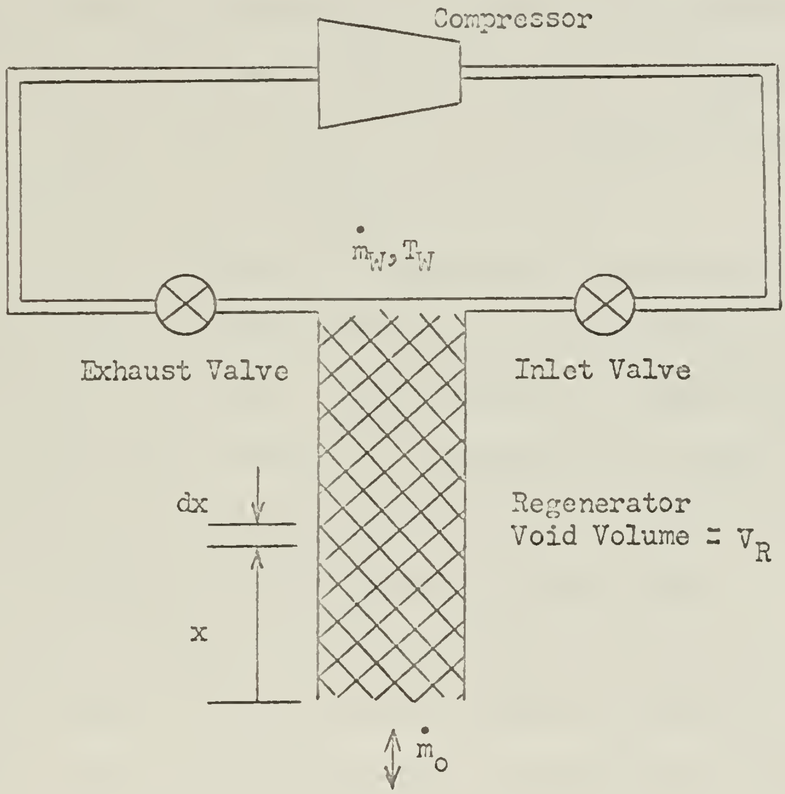
Previous Work

The first analysis of the pulse tube was given by Gifford and Longsworth in their first and second papers (1,2). This analysis did not predict compatible pulse tube capacity results with those they had obtained experimentally. Gifford and Longsworth recognized that their original analysis of the pulse tube was inadequate. Their original analysis neglected any heat exchange between the gas and the pulse tube wall. They discovered that this heat exchange could contribute a substantial amount to the refrigeration effect. In other words the pulse tube itself acts as an inefficient regenerator. The third paper by

Gifford and Longsworth (3) discussed the aspects of this heat exchange between the gas and the pulse tube wall in detail. However, no complete analysis including this effect was presented. The problem was primarily one of determining the appropriate temperature difference to use in

FIGURE II

SCHEMATIC OF REA'S 2-PART MODEL FOR A REGENERATOR (4)



order to calculate correctly the heat transferred in the process under consideration.

Concurrently with the work of Gifford and Longsworth, Rea was studying the performance of thermal regenerators subjected to rapid pressure and flow cycling (4). Rea's objective was to develop a theory for predicting the axial temperature distribution, mass flow rate

distribution, and effectiveness of thermal regenerators operated under the above conditions. In developing his theory Rea established his so-called 2-Part Model (4), which is patterned after the Gifford-McMahon cycle (5). A schematic of Rea's 2-Part Model is shown in Figure II.

By making energy and mass balances for the gas and the matrix in an elemental control volume, Rea was able to write the governing differential equations for his problem. After applying appropriate simplifications, Rea reduced his problem to a system of two differential equations, which he solved numerically with the aid of a computer. A complete description of Rea's 2-Part Model and the development of the governing differential equations are presented in Appendix A.

When the work of Gifford and Longsworth became known to him, Rea recognized that the Pulse Tube operates in the same manner as that described by his 2-Part Model. Rea concluded that since a pulse tube was nothing more than an inefficient regenerator, his theory should apply equally well to the entire Pulse Tube as to a simple regenerator. Thus, Rea extended his theory to include the pulse tube.

Rea tested his theory for the regenerator and got fairly accurate results. Rea, however, did not test his theory as extended to include the pulse tube. Thus, it is the purpose of this thesis to determine whether or not Rea's Theory correctly describes the performance of the Pulse Tube by, first, examining his theory and analysis for the entire Pulse Tube and by, second, measuring the refrigeration effect of an operating Pulse Tube.

CHAPTER II

EXAMINATION OF REA'S THEORY AS APPLIED TO A REGENERATOR AND "pulse tube"

Rea's 2-Part Model (4)

When faced with the task of formulating a mathematical analysis of a given system, the most important step is the selection of an appropriate model. Ideally one wants a model that is amenable to a theoretical treatment. On the other hand one does not want his model to be over-simplified, for then the problem may lose its significance. Rea selected as his model for the regenerator, and, hence, the pulse tube, a model that satisfies both of these criteria. This "2-Part Model", as Rea refers to it, is shown schematically in Figure II of Chapter I. The usefulness of this model is proved by the fairly accurate results which are obtained by its use and by the fact that certain dimensionless parameters which are definite aids in defining the problem evolve from its use.

The operation of the 2-Part Model is as follows: the inlet valve opens allowing high pressure gas to enter at the top of the device. During this half-cycle the pressure within the device continually increases with the mass flow rate out of the bottom of the device equal to $\dot{m}_o$. After a predetermined maximum pressure, p_{max}, is reached within the device, the inlet valve closes and the exit valve opens. During the exhaust half-cycle the process described above is reversed, with mass flow rate into the bottom of the device equal to $\dot{m}_o$. The pressure within the device decreases to

its minimum value, p_{min}, during this half-cycle. After a period of time, a steady, cyclic condition is reached.

In examining Rea's theory for the Pulse Tube, the theory for the individual components, i.e., the regenerator and pulse tube, will be examined separately.

Basic Differential Equations for the Regenerator and "pulse tube"

The governing differential equations are derived in detail in Appendix A by making a mass balance and energy balance for the gas and for the matrix in an elemental control volume, dx, at a position x within the regenerator and pulse tube respectively. The derivation of the regenerator equations is entirely attributable to Rea (4). The modification to Rea's Theory as applied to a pulse tube is original. The basic equations for the regenerator and pulse tube are:

Gas Energy Balance:

$$c_p \frac{\partial}{\partial x}(\dot{m}T_g) + hA_T(T_g - T) + \frac{c_v A_o}{R}\frac{\partial p}{\partial t} = 0 \quad , \qquad (2.1)$$

Matrix Energy Balance:

$$hA_T(T_g - T) = Mc_m \frac{\partial T}{\partial t} \quad , \qquad (2.2)$$

Gas Continuity:

$$\frac{\partial \dot{m}}{\partial x} = -\frac{A_o}{R}\frac{\partial}{\partial t}\left(\frac{p}{T_g}\right) = -\frac{A_o}{R}\left(\frac{1}{T_g}\frac{\partial p}{\partial t} - \frac{p}{T_g^2}\frac{\partial T_g}{\partial t}\right) \quad . \qquad (2.3)$$

The assumptions necessarily made in deriving equations (2.1), (2.2), and (2.3) are:

(1) Zero thermal conductivity in the direction of flow for both the gas and the matrix and infinite thermal conductivity in the matrix transverse to the flow direction,

(2) The pressure drop in the device is neglected,

(3) The gas obeys the perfect gas law,

(4) The gas transport properties and the matrix properties are constant,

(5) The cross sectional area of the device open to flow, A_o, is constant.

In addition to these assumptions it is necessary to assume that the heat transfer area per unit volume of the pulse tube is fairly large, in order that the pulse tube can be considered as a regenerator.

Boundary conditions for equations (2.1), (2.2), and (2.3) for the regenerator are:

$$\begin{aligned} \text{at } x = 0: \quad & \dot{m} = \dot{m}_o(t) \\ & T_g = T_{go}(t) \\ \text{at } x = L: \quad & T_g = T_{gW}(t)\ . \end{aligned} \tag{2.4}$$

Boundary conditions for equations (2.1), (2.2), and (2.3) for the pulse tube are:

$$\begin{aligned} \text{at } x = 0: \quad & \dot{m} = \dot{m}_o(t) \\ & T_g = T_{go}(t) \\ \text{at } x = L: \quad & T_g = T_{gC}(t) \end{aligned} \tag{2.5}$$

Note that these boundary conditions are identical and have been presented separately only to show that the position $x = L$ for the regenerator is the warm end of the regenerator, while the position $x = L$ for the pulse tube is the cold end of the pulse tube, when the Pulse Tube is operated as a refrigerator.

For a regenerator (and pulse tube in this case) operating in the steady, cyclic manner described here, certain cyclic conditions must be met. At a given position x, these conditions are as follows:

$$\oint hA_T(T_g - T)dt = Mc_m \frac{\partial T}{\partial t} dt = 0 \tag{2.6}$$

$$\oint \dot{m}dt = 0 \tag{2.7}$$

$$\oint \frac{\partial p}{\partial t} dt = 0 \tag{2.8}$$

$$\oint (\dot{m}T_g)dt = C_o \quad . \tag{2.9}$$

Equation (2.6) says that over a complete cycle there can be no net heat stored by the matrix. Equation (2.7) says that there is no net mass stored over a complete cycle. Equation (2.8) states that the pressure is cyclic. Equation (2.9) results from equations (2.6) and (2.8) applied to a cyclic integral of equation (2.1). C_o is constant with both time and position. The cyclic conditions, equations (2.6) to (2.9), are general in nature and must be satisfied by any steady, cyclic solution to equations (2.1), (2.2), and (2.3).

The complete cycle will now be broken into two half-cycles of equal time duration. Also, the pressure is assumed to be a sawtooth function of time such that $\left|\frac{dp}{dt}\right| =$ a constant. Other approximations and

assumptions are as follows:

(1) The temperature difference between the gas and the matrix at any point within the regenerator (or pulse tube) is very small. This enables one to substitute T for T_g in the final continuity relations.

(2) The product of the average equals the average of the product. This approximation is useful in applying the cyclic conditions, equations (2.6) through (2.9).

(3) For the regenerator only,

$$\frac{\partial}{\partial t}\left(\frac{p}{T_g}\right) \simeq \frac{1}{T}\frac{dp}{dt} \quad .$$

With these conditions imposed, the final set of equations to be solved for the regenerator are:

$$-c_p \frac{d}{dx}(\dot{m}T_{gc}) + hA_T(T_{gc} - T) + \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad , \tag{2.10}$$

$$c_p \frac{d}{dx}(\dot{m}T_{ge}) + hA_T(T_{ge} - T) - \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad , \tag{2.11}$$

$$\frac{d\dot{m}}{dx} = \frac{A_o}{RT}\left|\frac{dp}{dt}\right| \quad , \tag{2.12}$$

$$\dot{m}(T_{gc} - T_{ge}) = C_1 \quad , \tag{2.13}$$

$$T_{gc} + T_{ge} = 2T \quad . \tag{2.14}$$

The boundary conditions for these equations are:

$$\begin{aligned} \text{at } x = 0: \quad & T = T_o \\ & \dot{m} = \dot{m}_o \\ \text{at } x = L: \quad & T = T_W \end{aligned} \tag{2.15}$$

The final set of equations to be solved for the pulse tube are:

$$-c_p \frac{d}{dx}(\dot{m}T_{gc}) + hA_T(T_{gc} - T) + \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad , \tag{2.16}$$

$$c_p \frac{d}{dx}(\dot{m}T_{ge}) + hA_T(T_{ge} - T) - \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad , \tag{2.17}$$

$$\frac{d\dot{m}}{dx} = \frac{A_o}{RT}\left|\frac{dp}{dt}\right|\left[1 - 2C\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\Delta t}\,\frac{1}{T}\left(\frac{\dot{m}_o}{\dot{m}}\right)\left(\frac{V}{KA_T L}\right)\right], \tag{2.18}$$

$$\dot{m}(T_{gc} - T_{ge}) = C_1 \quad , \tag{2.19}$$

$$T_{gc} + T_{ge} = 2T \quad . \tag{2.20}$$

The boundary conditions for these equations are:

$$\begin{aligned} \text{at } x = 0: \quad & T = T_o \\ & \dot{m} = \dot{m}_o \\ \text{at } x = L: \quad & T = T_C \end{aligned} \tag{2.21}$$

Note that the set of equations for the regenerator is identical with the set of equations for the pulse tube with the exception that the continuity equations (2.12) and (2.18) are different. The difference between these two equations is that, in the case of the regenerator, the second term of equation (2.3) can be shown to be negligibly small in comparison to the first term, whereas in the case of the pulse tube, the second term is not small for all conditions of operation. Rea assumed that the continuity equation as written for the regenerator applied equally to the pulse tube. The details of the relative orders of magnitude of the terms of equation (2.3) for

the regenerator and pulse tube are presented in Appendix A.

Equations (2.10) through (2.14) are five equations in five unknowns for the regenerator and can be reduced to two equations in two unknowns by subtracting (2.11) from (2.10), and then by substituting into this result equations (2.12), (2.13), and (2.14), plus the commonly known relationship $R = c_p - c_v$. The result of this manipulation is:

$$\dot{m}\frac{dT}{dx} - \frac{hA_T C_1}{2c_p\dot{m}} + \frac{A_o}{c_p}\left|\frac{dp}{dt}\right| = 0 \quad , \tag{2.22}$$

and,

$$\frac{d\dot{m}}{dx} = \frac{A_o}{RT}\left|\frac{dp}{dt}\right| \quad . \tag{2.23}$$

The three boundary conditions (2.15) are required since, although C_1 in equation (2.22) is a constant, it is not known at this point.

Likewise, equations (2.16) through (2.20) are five equations in five unknowns for the pulse tube and in a similar fashion to that described above can be reduced to two equations in two unknowns. These results are:

$$\dot{m}\frac{dT}{dx} - \frac{hA_T C_1}{2c_p\dot{m}} + \frac{A_o}{c_p}\left|\frac{dp}{dt}\right|\left[\frac{\gamma\left(1 - 2C\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\Delta t}\,\frac{1}{T}\left(\frac{\dot{m}_o}{\dot{m}}\right)\left(\frac{V}{KA_T L}\right)\right) - 1}{\gamma - 1}\right] = 0 \tag{2.24}$$

and,

$$\frac{d\dot{m}}{dx} = \frac{A_o}{RT}\left|\frac{dp}{dt}\right|\left[1 - 2C\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\Delta t}\frac{1}{T}\left(\frac{\dot{m}_o}{\dot{m}}\right)\left(\frac{V}{KA_TL}\right)\right] \tag{2.25}$$

Again, the three boundary conditions (2.21) are required, since C_1 is unknown at this point.

Set-up of the Differential Equations for Computer Solution

The following dimensionless variables are now introduced:

$$\begin{aligned} \Theta &= T/T_o \\ m &= \dot{m}/\dot{m}_o \\ w &= \frac{V}{T_o\dot{m}_oR}\left|\frac{dp}{dt}\right|\frac{x}{L} \end{aligned} \tag{2.26}$$

Also it is assumed that the heat transfer coefficient can be correlated as:

$$h = Km^n \tag{2.27}$$

for the regenerator, and as:

$$h = K \tag{2.28}$$

for the pulse tube, which is assumed to be operating in laminar flow.

When the new variables (2.26), along with the assumption (2.27), are introduced into the regenerator equations (2.22) and (2.23), the final regenerator equations result in:

$$\frac{d\Theta}{dw} = \frac{C - \left(\frac{\gamma-1}{\gamma}\right)m^{1-n}}{m^{2-n}} \quad , \tag{2.29}$$

and $$\frac{dm}{dw} = \frac{1}{\Theta} \quad , \tag{2.30}$$

where $$C \equiv \frac{\gamma - 1}{\gamma} \left(\frac{KA_T L}{V} \right) \frac{C_1}{2\dot{m}_o} \frac{1}{\left| \frac{dp}{dt} \right|} \quad . \tag{2.31}$$

The boundary conditions for the regenerator become:

$$\text{at } w = 0: \quad m = 1$$
$$\Theta = 1$$

$$\text{at } w = w_{max} = \frac{V \left| \frac{dp}{dt} \right|}{T_o \dot{m}_o R} \quad : \quad \Theta = \frac{T_W}{T_o} = \Theta_W \quad . \tag{2.32}$$

In a similar fashion, when the new variables (2.26), along with the assumption (2.28), are introduced into the pulse tube equations (2.24) and (2.25), the final pulse tube equations result in:

$$\frac{d\Theta}{dw} = \frac{C \left(1 + \frac{\Phi}{\Theta} \right) - m \left(\frac{\gamma - 1}{\gamma} \right)}{m^2} \quad , \tag{2.33}$$

and $$\frac{dm}{dw} = \frac{1}{\Theta} \left(1 - \frac{\Phi C}{\Theta m} \right) \quad , \tag{2.34}$$

where C is defined by equation (2.31) and

$$\Phi \equiv 2 \left(\frac{\gamma}{\gamma - 1} \right) \left(\frac{V}{KA_T L} \right) \frac{p}{\Delta t} \frac{1}{T_o} \quad . \tag{2.35}$$

The boundary conditions for the pulse tube become:

$$\text{at } w = 0: \quad m = 1$$

$$\Theta = 1$$

$$\text{at } w = w_{max} = \frac{V}{T_o \dot{m}_o R} \frac{dp}{dt} \; : \; \Theta = \frac{T_C}{T_o} = \Theta_C. \tag{2.36}$$

Method of Solution

The method of solution for the regenerator equations (2.29) and (2.30) is to treat C as a parameter. This makes the problem an initial value problem, since both Θ and m are known at w = 0. To obtain a set of solutions, one must know the ratio of the specific heats, γ , and the type of matrix used in order that n can be assigned a value. The parameter C is then assigned a range of values, and the equations are numerically integrated from w = 0 to w = w_{max}. Solutions for n = .59 and γ = 1.67 are presented in graphical form in Appendix C.

The method of solution for the pulse tube equations (2.33) and (2.34) is similar to that for the regenerator. Now, however, there is a second parameter, $\bar{\Phi}$, involved. In this case $\bar{\Phi}$ is assigned a value, $\bar{\Phi} = \bar{\Phi}_1$, and then C is assigned a range of values for this $\bar{\Phi}$. Again the equations are numerically integrated from w = 0 to w = w_{max}. The result is a family of curves of Θ versus w and m versus w, with each curve having a specific value of C and each graph representing one value of the parameter $\bar{\Phi}$. Solutions for various values of $\bar{\Phi}$ and γ = 1.67 are presented in graphical form in Appendix C.

To use the curves in Appendix C one must know w_{max} and Θ_W for the case of the regenerator. In the case of the pulse tube, w_{max}, Θ_C, and $\bar{\Phi}$ must be known. Then entering the appropriate

graph with $w = w_{max}$ and $\Theta = \Theta_W$, or $\Theta = \Theta_C$, a value of C can be read. The axial temperature distribution is determined by following that value of C down to $w = 0$. Also, once C is known, the mass flow into the top of the regenerator (or the bottom of the pulse tube, as the case may be) is determined by entering the graph of m versus w at $w = w_{max}$ and reading off the value of m corresponding to the intersection of the C curve and the $w = w_{max}$ line. As in the case of the temperature distribution, the axial mass flow distribution is determined by following the same C curve from $w = w_{max}$ to $w = 0$. This yields the mass flow distribution from the warm end to the cold end of the regenerator or from the cold end to the warm end of the pulse tube.

From this point on equations (2.29), (2.30), (2.33), and (2.34), plus the boundary conditions equations (2.32) and (2.36), will be referred to as Rea's Modified Theory for the Pulse Tube.

CHAPTER III

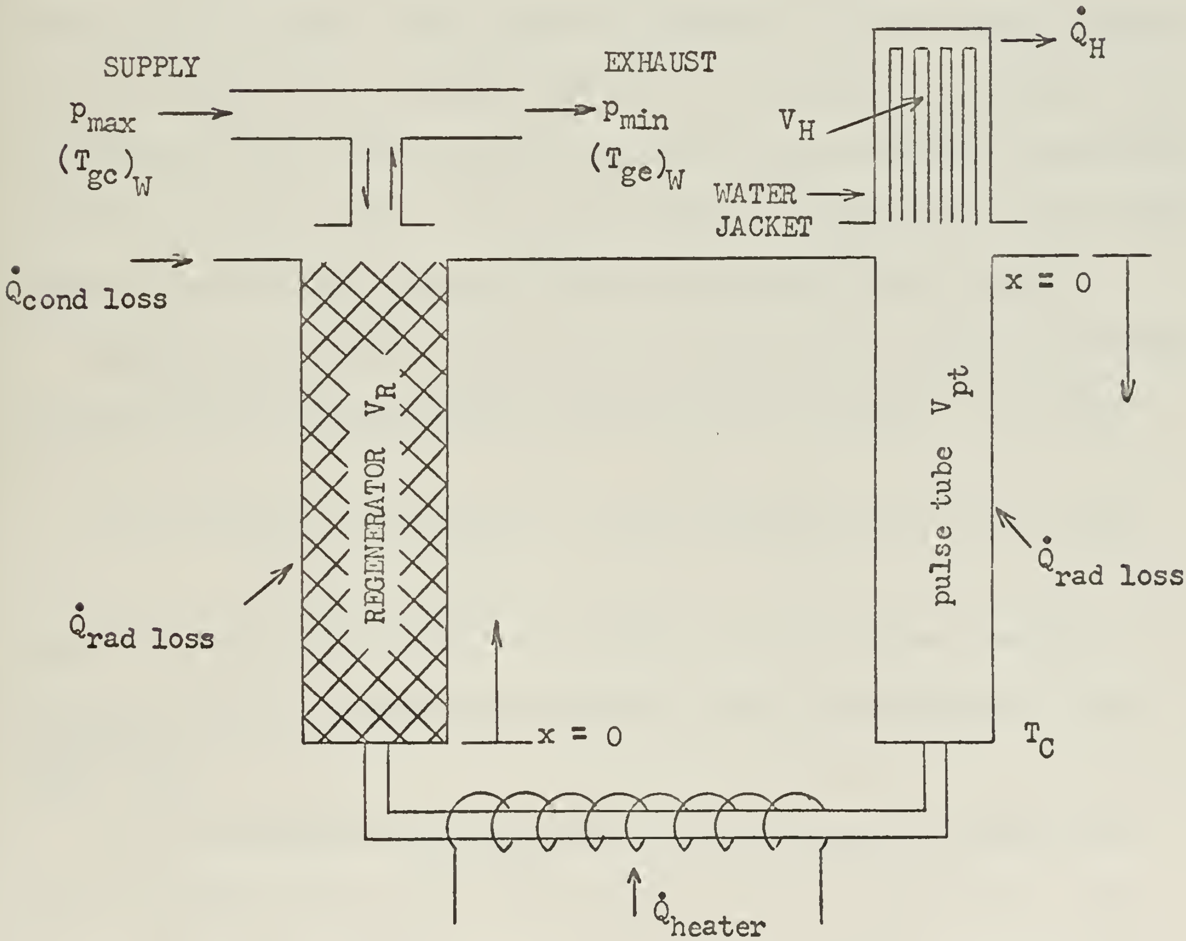
ANALYSIS OF THE PULSE TUBE

The System

The Pulse Tube consists of a regenerator in series with a pulse tube, as shown schematically in Figure III. This, then, is the system which will be analyzed.

FIGURE III

WORKING SCHEMATIC OF THE PULSE TUBE



Analysis

If one considers the control volume around the entire Pulse Tube as shown in Figure III and writes an energy balance for one cycle of length τ, the result is:

$$\tau_c c_p \left(\dot{m}_c T_{gc}\right)_W + \dot{Q}_{heater}\tau + \dot{Q}_{losses}\tau - \tau_e c_p \left(\dot{m}_e T_{ge}\right)_W$$

$$- \dot{Q}_H \tau = 0 \quad , \tag{3.1}$$

where the subscript W refers to the warm end of the regenerator. In equation (3.1) the first term represents the enthalpy carried by the gas into the control volume at the warm end of the regenerator, the second term is a known heat quantity introduced into the control volume, the third term is the heat associated with conduction and radiation that is absorbed by the control volume, the fourth term represents the enthalpy carried by the gas out of the control volume at the warm end of the regenerator, and the fifth term is the heat removed from the volume V_H.

Now, if the control volume around the pulse tube only is considered, one can write an energy balance for one cycle of length τ as follows:

$$\tau_c c_p \left(\dot{m}_c T_{gc}\right)_C + \dot{Q}'_{losses}\tau - \tau_e c_p \left(\mathring{m}_e T_{ge}\right)_C - \dot{Q}_H \tau = 0 \ , \tag{3.2}$$

where the subscript C refers to the cold end of the pulse tube and $\dot{Q}'_{losses}$ are the radiation and conduction losses associated with the pulse tube.

If one now makes the assumption that the period of compression, τ_c, is equal to the period of expansion, τ_e, and, furthermore, that either of these periods is equal to $\psi\tau$, where ψ is the percentage of

the total time that is devoted to compression or expansion, then equations (3.1) and (3.2) can be written as:

$$\psi c_p \left[\dot{m} \left(T_{gc} - T_{ge} \right) \right]_W + \dot{Q}_{heater} + \dot{Q}_{losses} - \dot{Q}_H = 0 \quad , \tag{3.1a}$$

and

$$\psi c_p \left[\dot{m} \left(T_{gc} - T_{ge} \right) \right]_C + \dot{Q}'_{losses} - \dot{Q}_H = 0 \quad . \tag{3.2a}$$

By combining equations (3.1a) and (3.2a) one obtains the result:

$$\dot{Q}_R = \psi c_p \left[\dot{m}_C \left(T_{gc} - T_{ge} \right)_C - \dot{m}_W \left(T_{gc} - T_{ge} \right)_W \right] \quad , \tag{3.3}$$

where

$$\dot{Q}_R \equiv \dot{Q}_{heater} + \sum \dot{Q}_{losses} \quad , \tag{3.4}$$

and $\dot{m}c_p \left(T_{gc} - T_{ge} \right)$ is the difference in enthalpy flux at some point in the regenerator or pulse tube. This enthalpy flux can be predicted by Rea's Theory as presented in Chapter II.

The difference in enthalpy flux at the cold end of the pulse tube and at the warm end of the regenerator will now be found.

From equation (2.31),

$$C_1 = 2C \left(\frac{\gamma}{\gamma - 1} \right) \left(\frac{V}{KA_T L} \right) \dot{m}_o \left| \frac{dp}{dt} \right| \quad , \tag{3.5}$$

and from equation (2.13),

$$C_1 = \dot{m} \left(T_{gc} - T_{ge} \right) \quad . \tag{3.6}$$

Now consider the regenerator. At the bottom, or cold end, $x = 0$ and $\dot{m} = \dot{m}_o = \dot{m}_C$. At the top, or the warm end, $x = L$, $T = T_W$, and $\dot{m} = \dot{m}_W$. From equation (3.6),

$$\dot{m}_W \left(T_{gc} - T_{ge} \right)_W = \left(C_1 \right)_W \equiv \left(C_1 \right)_R \quad . \tag{3.7}$$

Then from equation (3.5),

$$\left(C_1 \right)_R = 2C_R \left(\frac{\gamma}{\gamma - 1} \right) \left(\frac{V}{KA_T L} \right)_R \dot{m}_C \left| \frac{dp}{dt} \right| \quad . \tag{3.8}$$

Now consider the pulse tube. At the connection between the cooler section V_H and the pulse tube, $x = 0$ and $\dot{m} = \dot{m}_o = \dot{m}_H$. At the bottom, or the cold end, $x = L$, $\dot{m} = \dot{m}_L = \dot{m}_C$, and $T = T_L = T_C$. From equation (3.6),

$$\dot{m}_C \left(T_{gc} - T_{ge} \right)_C = \left(C_1 \right)_C \equiv \left(C_1 \right)_{pt} \quad . \tag{3.9}$$

Then from equation (3.5),

$$\left(C_1 \right)_{pt} = 2C_{pt} \left(\frac{\gamma}{\gamma - 1} \right) \left(\frac{V}{KA_T L} \right)_{pt} \dot{m}_H \left| \frac{dp}{dt} \right| \quad . \tag{3.10}$$

By substituting equations (3.7) and (3.9) into equation (3.3) one obtains:

$$\dot{Q}_R = \psi \, c_p \left[\left(C_1 \right)_{pt} - \left(C_1 \right)_R \right] \quad . \tag{3.11}$$

Now by substituting equations (3.8) and (3.10) into equation (3.11) the result is:

$$\dot{Q}_R = 2\,\Psi\, c_p \left(\frac{\gamma}{\gamma-1}\right) \left|\frac{dp}{dt}\right| \left[\dot{m}_H \left(\frac{V}{KA_T L}\right)_{pt} C_{pt} - \dot{m}_C \left(\frac{V}{KA_T L}\right)_R C_R\right]. \quad (3.12)$$

Equation (3.12) is identical with equation (3.4) of Rea (4) with the exception of the Ψ factor, which apparently was omitted by Rea. The method of determining this factor will be commented on in the following chapter.

Equation (3.12) will now be simplified. One is able to write for a perfect gas:

$$pV = mRT_g \quad (3.13)$$

where m is the mass. If this equation is differentiated with respect to time for a particular volume, the result is the general relationship:

$$\dot{m} = \frac{V}{R}\left(\frac{1}{T_g}\ \frac{dp}{dt} - \frac{p}{T_g^2}\ \frac{dT_g}{dt}\right) \quad . \quad (3.14)$$

Now consider the particular volume to be V_H, the volume of the cooler at the end of the pulse tube. The assumption is now made that there is perfect heat transfer in V_H. This means that the gas in the volume V_H is at a constant temperature T_H. With this assumption equation (3.14) becomes:

$$\dot{m}_H = \frac{V_H}{RT_H} \left| \frac{dp}{dt} \right| \quad . \tag{3.15}$$

From equation (3.15) it follows that

$$\dot{m}_C = \frac{\dot{m}_C}{\dot{m}_H} \frac{V_H}{RT_H} \left| \frac{dp}{dt} \right| \quad . \tag{3.16}$$

By substituting equations (3.15) and (3.16) into equation (3.12) and by rearranging, equation (3.12) can be written:

$$\dot{Q}_R = 2\psi \left(\frac{\gamma}{\gamma - 1} \right)^2 \left| \frac{dp}{dt} \right|^2 \frac{V_H}{T_H} \left(\frac{\epsilon}{K\beta} \right)_{pt} \left(C_{pt} - \alpha C_R \right) \tag{3.17}$$

where

$$\alpha \equiv \frac{\dot{m}_C}{\dot{m}_H} \frac{\epsilon_R}{\epsilon_{pt}} \frac{(K\beta)_{pt}}{(K\beta)_R} \tag{3.18}$$

and

$$\beta \equiv \frac{A_T L \epsilon}{V} \quad . \tag{3.19}$$

Equation (3.17) is the governing equation for the Pulse Tube.

CHAPTER IV

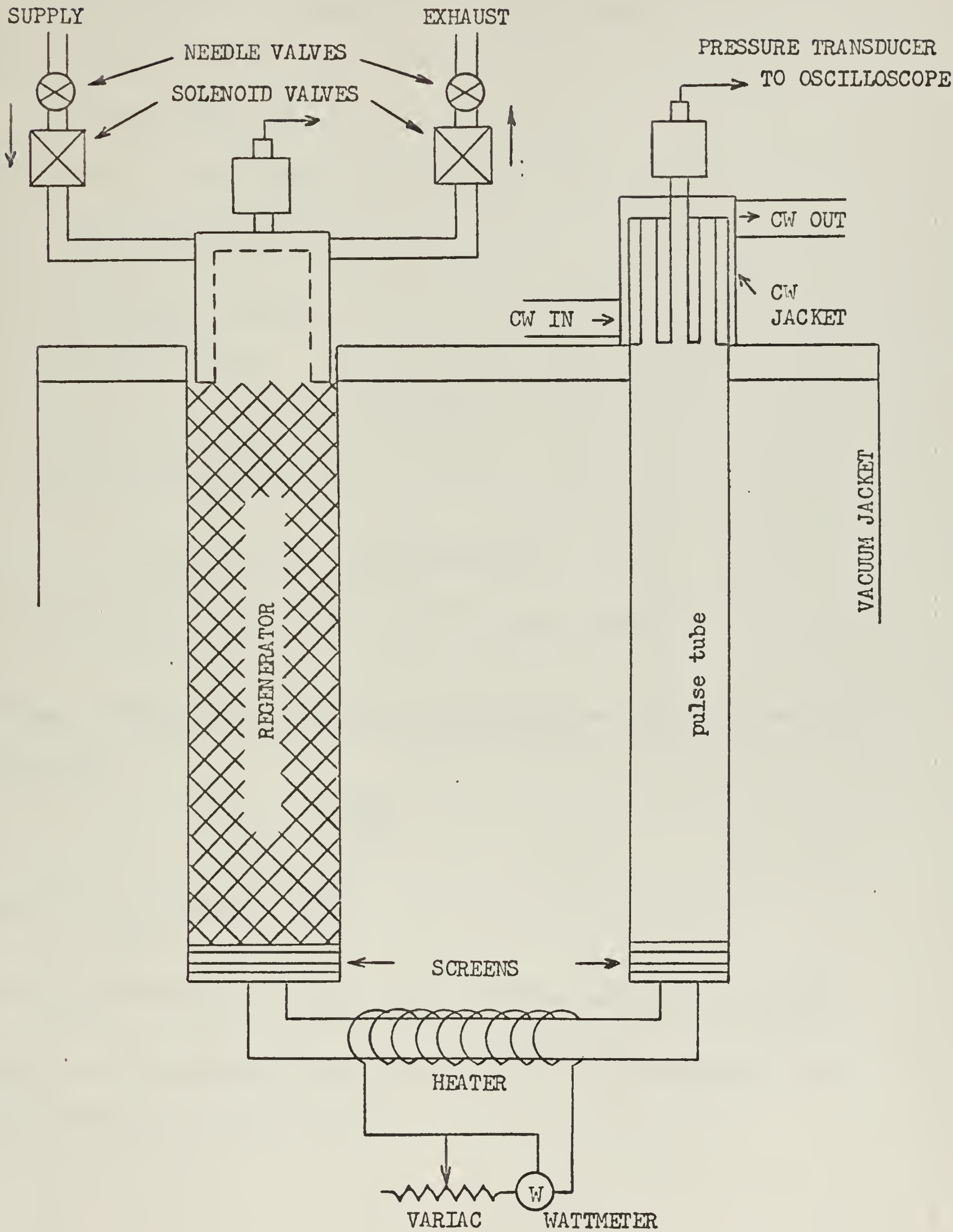
EXPERIMENTAL PROCEDURE

Description of Apparatus

Rea's Modified Theory for the Pulse Tube was tested using an apparatus constructed as shown schematically in Figure IV. The regenerator consisted of a 12"-long, stainless-steel tube, 1" O.D. by 0.020"-wall-thickness, packed with 0.050"-diameter lead shot. The pulse tube consisted of a 12"-long, stainless-steel tube, 5/8" O.D. by 0.016"-wall-thickness. The bottom of the regenerator was connected to the bottom of the pulse tube by a 1/4" O.D. by 0.020"-wall-thickness stainless-steel tube. Fine-mesh-bronze screens were installed at the bottom of the regenerator and pulse tube to smooth the flow. The heat exchanger at the top of the pulse tube was constructed from thin-wall 1/8"-copper-nickel tubing. The rate of flow of cooling water around these tubes was 0.5 gal/min. A heating coil of high resistance wire was wound around the connecting tubing between the regenerator and pulse tube as a means of introducing a known heat load into the device. Copper-constantan thermocouples were attached along the length of the regenerator and pulse tube. Pressure transducers were installed at the warm end of the regenerator and at the water-cooler end of the pulse tube. The entire Pulse Tube was suspended in a vacuum-tight chamber for thermal insulation. During data runs this vacuum was maintained at approximately 2 x 10^{-5} mm. Hg. Helium was used as the working gas for the Pulse Tube. A four-speed motor allowed a choice of cycling rates corresponding to 22.4, 44.7, 82.3, and 158.0

FIGURE IV

SCHEMATIC OF TEST APPARATUS



reversals per minute. A complete description of the apparatus is contained in Appendix D.

The important characteristics of the device were:

(1) $V_R = 3.27 \text{ in}^3$

(2) $V_{pt} = 3.39 \text{ in}^3$

(3) $V_H = 0.673 \text{ in}^3$

(4) $V_{pt}/V_H = 5.04$

(5) $V_R/V_{pt} = 0.964$

(6) $\epsilon_R = 0.388$

(7) $\epsilon_{pt} = 1.0$

(8) $\gamma = 1.67$

(9) $(K\beta)_{pt} = 595\ k \quad \text{BTU/in-hr-ft}^2\text{-}^\circ\text{F}$

(10) $(K\beta)_R = 1.14 \times 10^5\ (\dot{m}_C)^{.59} \quad \text{BTU/in-hr-ft}^2\text{-}^\circ\text{F}$

$(K\beta)_{pt}$, assuming fully developed laminar flow, was obtained from the expression:

$$(K\beta)_{pt} = \frac{17.44K}{d_h{}^2}$$

where Nu = 4.36 and $\beta = \dfrac{A_T L \epsilon}{V}$.

$(K\beta)_R$ was obtained as follows: for a spherical matrix, $\beta_R = \dfrac{6(1-\epsilon)}{d}$,

where d is the diameter of the matrix spheres. K_R was determined from Rea's experimental results (4).

Method of Testing

Basically the objective of the experimental work was to obtain data from a Pulse Tube that could be compared with the refrigeration effect which Rea's Modified Theory predicts for that Pulse Tube.

One of the most important experimental results was the axial temperature distribution, particularly of the pulse tube. From the equation (2.26), w is defined as:

$$w \equiv \frac{V}{T_o \dot{m}_o R} \left| \frac{dp}{dt} \right| \frac{x}{L} \quad . \tag{4.1}$$

For the pulse tube, from equation (3.15):

$$\dot{m}_o = \dot{m}_H = \frac{V_H}{T_H R} \left| \frac{dp}{dt} \right| \tag{4.2}$$

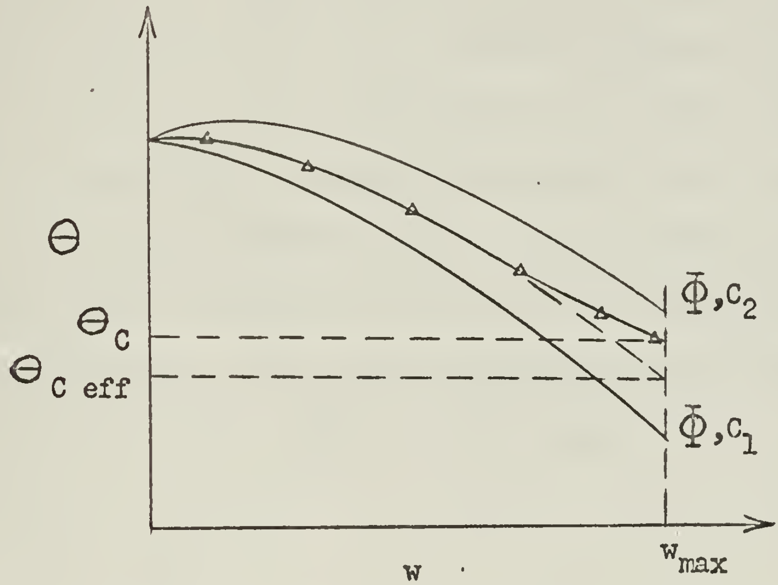
Therefore, for the pulse tube:

$$w_{pt} = \frac{V_{pt}}{V_H} \frac{x}{L_{pt}} \quad , \tag{4.3}$$

where x is the distance along the pulse tube measured from the top as shown in Figure III. For a given pulse tube, equation (4.3) allows one to specify locations along its length as a function of w_{pt}. This was helpful in that thermocouples could be located at any distance x along the pulse tube and that position could be translated easily into a corresponding w_{pt}.

The assumption was made that the temperature of the gas inside the cooler was uniform and at the same value as that of the cooling water. In reality this is not possible, but by using thin-wall cooler tubes and by allowing a short dwell time between the compression and expansion

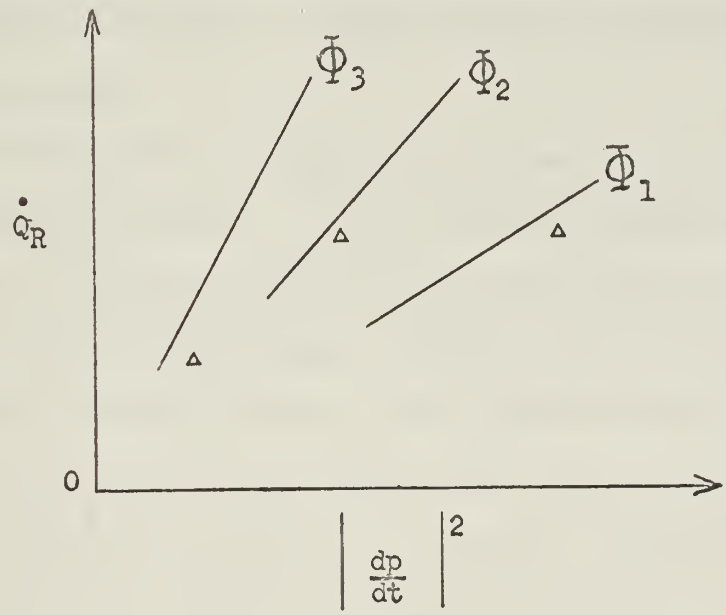
parts of the cycle, "perfect" heat transfer could be approached. In this case $(T_o)_{pt}$ became simply T_H. Then by measuring the temperature of the pulse tube at each thermocouple location and by dividing each of these temperatures by T_H to convert to the non-dimensional temperature variable Θ, the experimental temperature distribution of the pulse tube could be determined readily. When this experimental distribution was compared with theoretical temperature distributions, one obtained typically:



The shape of all the experimental curves was similar to the shape of the theoretical curves except in the vicinity of w_{max}, indicating that the laminar flow assumption for the pulse tube was fairly accurate. In normal application the theoretical curves would be entered knowing a value of w_{max} and Θ_C. However, in this case, since the entire pulse tube temperature distribution was known, it was more appropriate to use $(\Theta_C)_{eff}$ as an entering argument. This gave a more accurate value of C_{pt} than if Θ_C had been used.

The selection of the method of comparing the calculated results

with the experimental results will now be discussed. From equation (3.4) it should be noted that $\mathring{Q}_R$ includes the input power from the heater plus the conduction and radiation losses in the regenerator and pulse tube. Since both conduction and radiation losses depend upon the axial temperature distribution, these losses would be approximately constant for any operating conditions of the device which resulted in the same temperature distribution. This suggested a possible procedure for testing wherein a change in the measured value of $\dot{Q}_R(\dot{Q}_{heater} + \sum \dot{Q}_{losses})$ could be compared with a change in the calculated value of $\dot{Q}_R$ for similar temperature distributions and for identical changes in $\left|\frac{dp}{dt}\right|^2$. The $\dot{Q}_{losses}$, of course, had to be determined. As a first approximation they were analytically obtained from equation (3.17) for no heater input. This approximation assumed that the analytical expression is correct. This assumption may not be exact, but should suffice as a first approximation. The comparisons suggested above could be made on a plot of $\dot{Q}_R$ versus $\left|\frac{dp}{dt}\right|^2$ as shown below:



A smooth curve cannot be legitimately drawn through the measured points of the $\dot{Q}_R$ versus $\left|\frac{dp}{dt}\right|^2$ plot, since there is a third variable, Φ , which was not fixed. Another method of conducting the experiment would have been to keep Φ constant, and then compare changes in experimental $\dot{Q}_R$ with changes in predicted $\dot{Q}_R$.

Based on the above discussion an experimental procedure was selected as follows: First, a cycling speed was arbitrarily selected from the four speeds available. Second, a Θ_C was arbitrarily chosen. The selection of Θ_C had one limitation, that being the value selected had to be greater than the lowest possible Θ_C obtainable with the device under the limiting conditions of operation selected. This limitation was necessary so that a range of operation was available wherein the same Θ_C could be maintained with an increasing heat load. Third, a ratio of p_{max}/p_{min} was selected which would maintain the chosen Θ_C. Initially there was no heater input. Under these conditions the refrigeration effect consisted of conduction and radiation losses only. It took 60-90 minutes for the temperature distribution to become steady. Once steady state was reached, the $\left|\frac{dp}{dt}\right|$ required to maintain the selected Θ_C was noted.

Next, the heater power and $\left|\frac{dp}{dt}\right|$ were increased in such a manner that the Θ_C remained constant. It was found that response time to slight changes in pressures was considerably less than the response time to changes in the heater input. Therefore, for rapid stabilization, the heater power was increased a fixed amount and then the $\left|\frac{dp}{dt}\right|$ was adjusted to achieve the desired Θ_C. Once heater power was introduced, it was impossible to maintain the identical regenerator temperature distribution, although Θ_W did remain nearly constant. This, of

course, infers that the $\dot{Q}_{losses}$ were not exactly constant for a series of data runs. The temperature distribution of the pulse tube on the other hand could be very closely duplicated. Once the steady condition was achieved, the $\left|\frac{dp}{dt}\right|$ and $\dot{Q}_{heater}$ were recorded. By adding the $\dot{Q}_{losses}$ to the $\dot{Q}_{heater}$, the $\dot{Q}_R$ was obtained. By continuing this process a series of experimental points of $\dot{Q}_R$ versus $\left|\frac{dp}{dt}\right|^2$ could be obtained.

In the above procedure $\dot{Q}_{heater}$ was determined by measuring wattage and presented no difficulty. Because of the strong dependency of the Pulse Tube performance on $\left|\frac{dp}{dt}\right|$, provisions were made to measure the pressure at both the warm end of the regenerator and water-cooler end of the Pulse Tube. To check the pressure drop through the Pulse Tube, the water-cooler end pressure transducer was first calibrated under static conditions using previously calibrated pressure gauges which were installed at the warm end of the regenerator as a reference. The pressure was then allowed to cycle. Photographs of the oscilloscope traces of the pressure tranducer output were taken for different pressure settings on the gauges. From these photographs and the pressure transducer calibration curve, the maximum and minimum pressures at the water-cooler end of the pulse tube were obtained. For most pressure settings the measurements taken at both ends of the Pulse Tube were essentially identical. For cycling speeds up to and including 82.3 reversals per minute, the only measurable pressure drop for the data taken was approximately 2 psi at a supply pressure of 150 psig. For these runs it made no difference where in the Pulse Tube the pressure measurements were taken. With the highest speed of 158.0 reversals, the pressure drop ranged from 3 psi at 50 psig to 24 psi at 112 psig. For these runs the p_{max} and p_{min} were measured on the pulse

tube side of the device.

Typical pressure transducer traces are shown in Figure V. From the traces it can be seen that the total period can be obtained precisely. However, the values for τ_c, τ_e, and $\left|\frac{dp}{dt}\right|$ are not quite so obvious. In the analysis of the Pulse Tube it was assumed that $\tau_c = \tau_e = \psi\tau$. In keeping with this assumption the following values of $\psi\tau$ were decided upon by taking an arithmetic mean of τ_c and τ_e:

Reversals/Min	$\psi\tau$(sec)	ψ
22.4	1.0	0.372
44.7	0.5	0.372
82.3	0.27	0.372
158.0	0.14	0.372

Because of the difference in slopes of the compression and expansion parts of the cycle and the slight non-linearity of the exhaust history, average values were also used to determine $\left|\frac{dp}{dt}\right|$. With a similar pressure history, Rea (4) reported good results by taking:

$$\left|\frac{dp}{dt}\right|_{avg} = \frac{p_{max} - p_{min}}{\Delta t_{avg}} , \tag{4.5}$$

where

$$\Delta t_{avg} = \psi\tau = \tfrac{1}{2}\left(\tau_c + \tau_e\right) \tag{4.6}$$

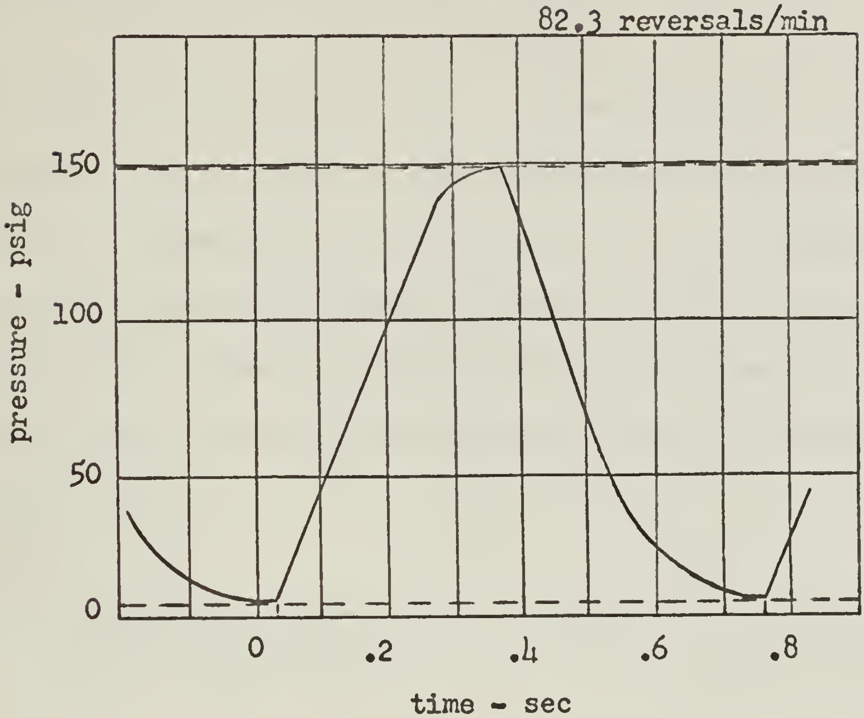
and p_{max} = maximum cycle pressure

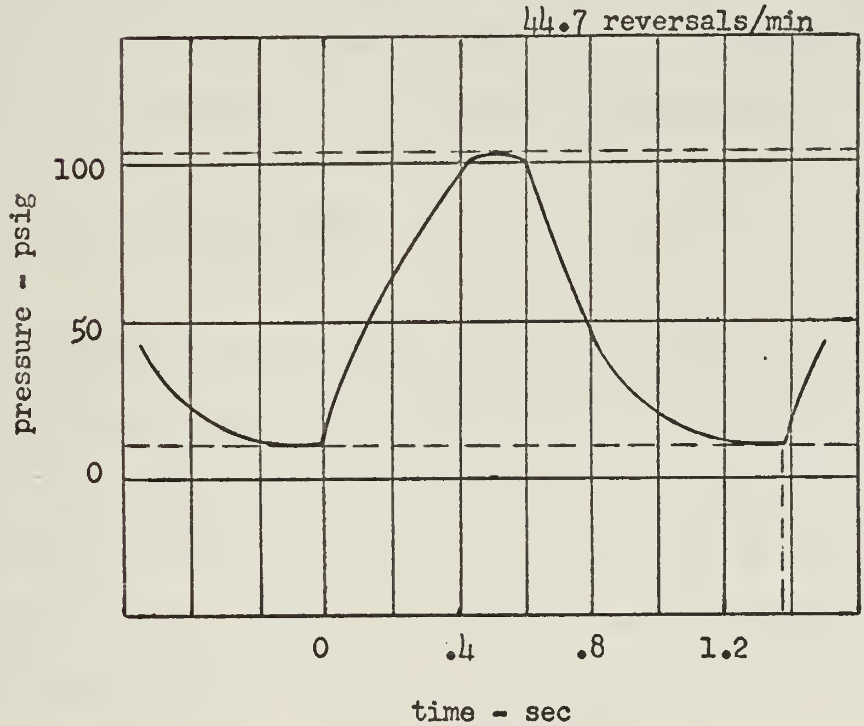
p_{min} = minimum cycle pressure.

A similar relationship for $\left|\frac{dp}{dt}\right|$ was used in this case.

FIGURE V

TYPICAL "pulse tube" PRESSURE - TIME HISTORIES





To determine a predicted value for $\dot{Q}_R$, equation (3.17) was used:

$$\dot{Q}_R = 2\,\psi\left(\frac{\gamma}{\gamma-1}\right)^2 \left|\frac{dp}{dt}\right|^2 \frac{V_H}{T_H}\left(\frac{\epsilon}{K\beta}\right)_{pt}\left(C_{pt} - \alpha C_R\right) \quad ,$$

where α is defined by equation (3.18). For any Pulse Tube ϵ_{pt}, ϵ_R, and V_H would be known. γ and T_H would also be known depending on the gas used and the temperature of the cooling water. ψ and $\left|\frac{dp}{dt}\right|$ were determined by the methods previously discussed. $(K\beta)_{pt}$ was known once the pulse tube temperature distribution became known. To determine a value of C_{pt} a value of the parameter Φ had to be determined. From equation (2.35):

$$\Phi \equiv 2\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{V}{KA_T L}\right)\frac{p}{\Delta t}\,\frac{1}{T_o} \quad .$$

In the above equation γ and $\left(\frac{V}{KA_T L}\right)$ were known; T_o corresponds to T_H and was also known; Δt was taken as $\psi\tau$, which was obtained from equation (4.6); p was taken as p_{avg}, which was defined as:

$$p_{avg} = p_{min} + \frac{p_{max} - p_{min}}{2} \quad . \tag{4.7}$$

Thus Φ was obtained. Now since Φ, $(w_{max})_{pt}$ and $(\Theta_C)_{eff}$ were all known, the appropriate plot of Θ versus w for the pulse tube was entered and a C_{pt} extracted. This in turn led to a value of $\dot{m}_C/\dot{m}_H$ by the procedure described in Chapter II. From $\dot{m}_C/\dot{m}_H$, $\dot{m}_C$ was found. This led to a value for $(K\beta)_R$. The next step was to determine $(w_{max})_R$, which was obtained from:

$$(w_{max})_R = \frac{\frac{T_H}{T_C} \frac{V_R}{V_H}}{\frac{\dot{m}_C}{\dot{m}_H}} \quad . \tag{4.8}$$

Equation (4.8) follows from the definition of w_{max}, equation (2.26), assuming that $\dot{m}_C$ is constant along the connecting tubing. All quantities in equation (4.8), then, were known. Now knowing Θ_W and $(w_{max})_R$, a value of C_R was extracted from the regenerator curves of Θ versus w. From equation (3.18) α was calculated, since all necessary quantities were known. Finally, $\dot{Q}_R$ was calculated from equation (3.17).

Sample Calculation

Suppose one wants to predict the refrigeration for a Pulse Tube with the following specifications:

T_H = cooling water temperature = $502^{\circ}R$

$T_W = 477^{\circ}R$

$V_{pt}/V_H = 0.673\ in^3$

$V_R = 3.27\ in^3$

Refrigeration is desired at $376^{\circ}R$. Hence, $T_C/T_H = 0.75$ and $T_W/T_C = 1.27$. Helium is to be used as the working gas. The cycle rate is to be 82.3 reversals per minute. The pressures to be used are:

$p_{max} = 60$ psig

$p_{min} = 10$ psig

$p_{atm} = 15$ psia

Φ is calculated from equation (2.35) to be 2.46. From Figures XVIII through XXVIII in Appendix C with $w_{max} = 5.0$ and $T_C/T_H = 0.75$, a cross curve of Φ versus C_{pt} can be obtained. Entering this cross curve with $\Phi = 2.46$, a value of $C_{pt} = 0.147$ is read. In an analogous manner a cross curve of m versus Φ can be obtained. From this curve $\dot{m}_C/\dot{m}_H = 5.8$ is read. Then, from equation (4.8),

$$(w_{max})_R = 1.14 \quad .$$

Now from Figure XVI of Appendix C with $w = 1.14$ and $T_W/T_C = 1.27$, $C_R = 0.85$ is read. Suppose also that α has been calculated to be 0.00811., then $\alpha C_R = 0.0069$. For a cycle rate of 82.3, $\psi\tau = 0.27$ sec and $\psi = 0.372$. From equation (4.5):

$$\left| \frac{dp}{dt} \right| = 185 \text{ psi/sec} \quad .$$

In addition $(K\beta)_{pt} = 41.6$ BTU/in-hr-ft^2-$^\circ$F.

Then, from equation (3.17):

$$\dot{Q}_R = 4.25 \text{ watts}$$

CHAPTER V

RESULTS

Presentation of Results

Experimental data was obtained using the apparatus and the procedure described in the preceeding chapter. Three series of data runs were made. In the first series different speeds and widely variant back pressures were used for the various runs. For all runs in the second and third series a single speed and a nearly constant back pressure was used. Tables I, II, and III summarize the results for these series of data. Figures VI through VIII illustrate the comparisons between the predicted and the actual refrigeration loads. Observed and theoretical pulse tube axial temperature distributions for the three series of runs are presented in Figures IX and X. Figure XI shows a plot of m and C_{pt} versus Φ for $w = (w_{max})_{pt}$ and $\Theta = (\Theta_C)_{eff}$.

TABLE I

EXPERIMENTAL AND THEORETICAL RESULTS - SERIES 1

Run	1-1	1-2	1-3	1-4	1-5
DATA:					
T_C (^oR)	391.7	391.7	391.7	391.7	391.7
T_H (^oR)	498.7	498.7	498.7	498.7	498.7
$(T_C/T_H) = \Theta_C$	0.788	0.788	0.788	0.788	0.788
$(T_W/T_C) = \Theta_W$	1.25	1.22	1.27	1.25	1.27
Cycle Rate (reversals/min)	44.7	44.7	22.4	44.7	82.3
p_{max} (psig)	100	180	200	160	145
p_{min} (psig)	30	15	40	50	60
Pressure Drop (psi)	0	0	0	0	2
Heater Power (watts)	0	5.6	0.86	1.48	1.45
CALCULATED RESULTS:					
$(\Theta_C)_{eff}$	0.77	0.77	0.77	0.77	0.77
dp/dt (psi/sec)	140	330	160	220	315
$(dp/dt)^2 \times 10^{-4}$ $(psi/sec)^2$	1.96	10.9	2.56	4.84	9.9
p_{avg} (psia)	80	112	135	120	117
Φ	2.15	3.00	1.81	3.22	5.80
C_{pt}	0.173	0.136	0.195	0.128	0.080
$m_{pt} = \dot{m}_C/\dot{m}_H$	5.64	5.55	5.70	5.51	5.4
$(w_R)_{max}$	1.12	1.13	1.10	1.14	1.17
C_R	0.84	0.83	0.86	0.84	0.85
αC_R	0.008	0.005	0.007	0.006	0.005
Total Calculated Refrigeration (watts)	2.15	9.42	3.18	3.90	4.92

TABLE II

EXPERIMENTAL AND THEORETICAL RESULTS - SERIES 2

Run	2-1	2-2	2-3	2-4	2-5	2-6	2-7
DATA:							
T_C (^oR)	375.7	375.7	375.7	375.7	375.7	375.7	375.7
T_H (^oR)	502.2	502.2	502.2	502.2	502.2	502.2	502.2
$(T_C/T_H) = \Theta_C$	0.750	0.750	0.750	0.750	0.750	0.750	0.750
$(T_W/T_C) = \Theta_W$	1.27	1.24	1.24	1.21	1.21	1.20	1.19
Cycle Rate (reversals/min)	82.3	82.3	82.3	82.3	82.3	82.3	82.3
p_{max} (psig)	60	72	90	108	130	140	153
p_{min} (psig)	10	8	12	10	10	10	10
Pressure Drop (psi)	0	0	0	0	0	2	2
Heater Power (watts)	0	1.84	3.59	6.0	8.56	11.4	13.9
CALCULATED RESULTS:							
$(\Theta_C)_{eff}$	0.735	0.735	0.735	0.735	0.735	0.735	0.735
dp/dt (psi/sec)	185	237	289	366	444	482	530
$(dp/dt)^2 \times 10^{-4}$ $(psi/sec)^2$	3.43	5.60	8.32	13.4	19.7	23.2	28.0
p_{avg} (psia)	50	55	66	74	85	90	97
Φ	2.46	2.71	3.25	3.64	4.18	4.43	4.78
C_{pt}	0.147	0.139	0.120	0.108	0.096	0.091	0.086
$m_{pt} = \dot{m}_C/\dot{m}_H$	5.8	5.8	5.75	5.75	5.75	5.75	5.75
$(w_R)_{max}$	1.14	1.14	1.15	1.15	1.15	1.15	1.15
C_R	0.85	0.82	0.80	0.78	0.78	0.75	0.75
αC_R	0.007	0.006	0.005	0.004	0.004	0.003	0.003
Total Calculated Refrigeration (watts)	4.25	6.7	8.5	12.3	16.1	18.3	20.6

TABLE III

EXPERIMENTAL AND THEORETICAL RESULTS - SERIES 3

Run	3-1	3-2	3-3	3-4	3-5
DATA:					
T_C (^oR)	363.2	363.2	363.2	363.2	363.2
T_H (^oR)	507.7	507.7	507.7	507.7	507.7
$(T_C/T_H) = \Theta_C$	0.714	0.714	0.714	0.714	0.714
$(T_W/T_C) = \Theta_W$	1.36	1.35	1.32	1.31	1.34
Cycle Rate (reversals/min)	158	158	158	158	158
P_{max} (psig)	50	56	74	85	112
P_{min} (psig)	13	10	13	13	22
Pressure Drop (psi)	3	6	7	16	24
Heater Power (watts)	0	2.56	5.0	10.0	15.5
CALCULATED RESULTS:					
$(\Theta_C)_{eff}$	0.70	0.70	0.70	0.70	0.70
dp/dt (psi/sec)	264	328	435	514	642
$(dp/dt)^2 \times 10^{-4}$ (psi-sec)^2	7.0	10.8	19.0	26.4	41.0
P_{avg} (psia)	47	48	58	64	82
Φ	4.41	4.50	5.45	6.0	7.7
C_{pt}	0.084	0.084	0.071	0.065	0.052
$m_{pt} = \dot{m}_C/\dot{m}_H$	6.0	6.0	5.95	5.95	5.9
$(w_R)_{max}$	1.15	1.15	1.16	1.16	1.17
C_R	0.98	0.96	0.93	0.92	0.95
αC_R	0.006	0.005	0.005	0.004	0.004
Total Calculated Refrigeration (watts)	4.96	7.64	11.4	14.7	17.9

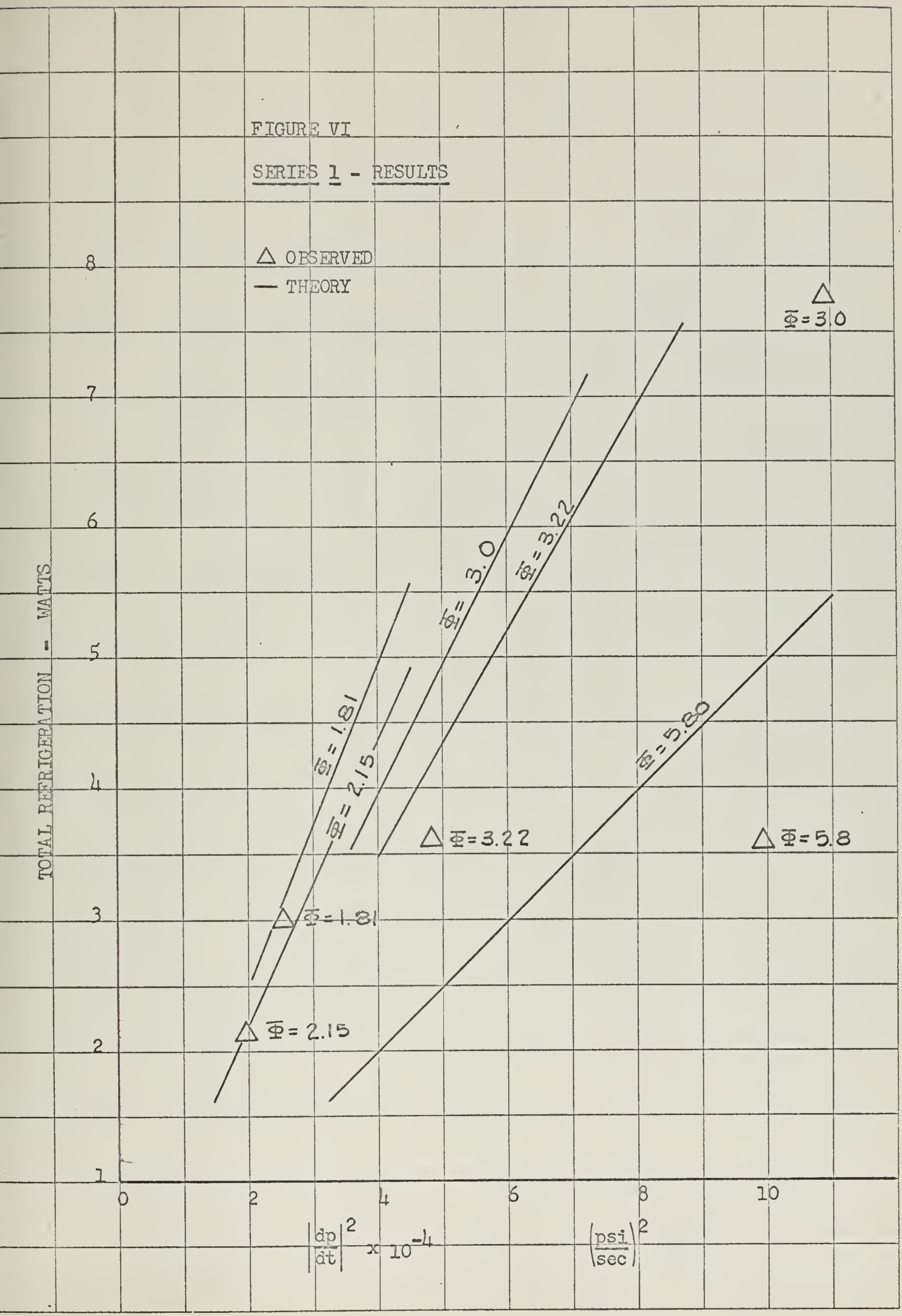


FIGURE VI
SERIES 1 - RESULTS
△ OBSERVED
— THEORY
TOTAL REFRIGERATION - WATTS
8
7
6
5
4
3
2
1
0
2
4
6
8
10
$\left|\frac{dp}{dt}\right|^2 \times 10^{-4}$
$\left(\frac{psi}{sec}\right)^2$
$\Phi = 3.0$
$\Phi = 3.22$
$\Phi = 1.81$
$\Phi = 2.15$
$\Phi = 5.80$
$\Phi = 3.22$
$\Phi = 5.8$
$\Phi = 1.81$
$\Phi = 2.15$

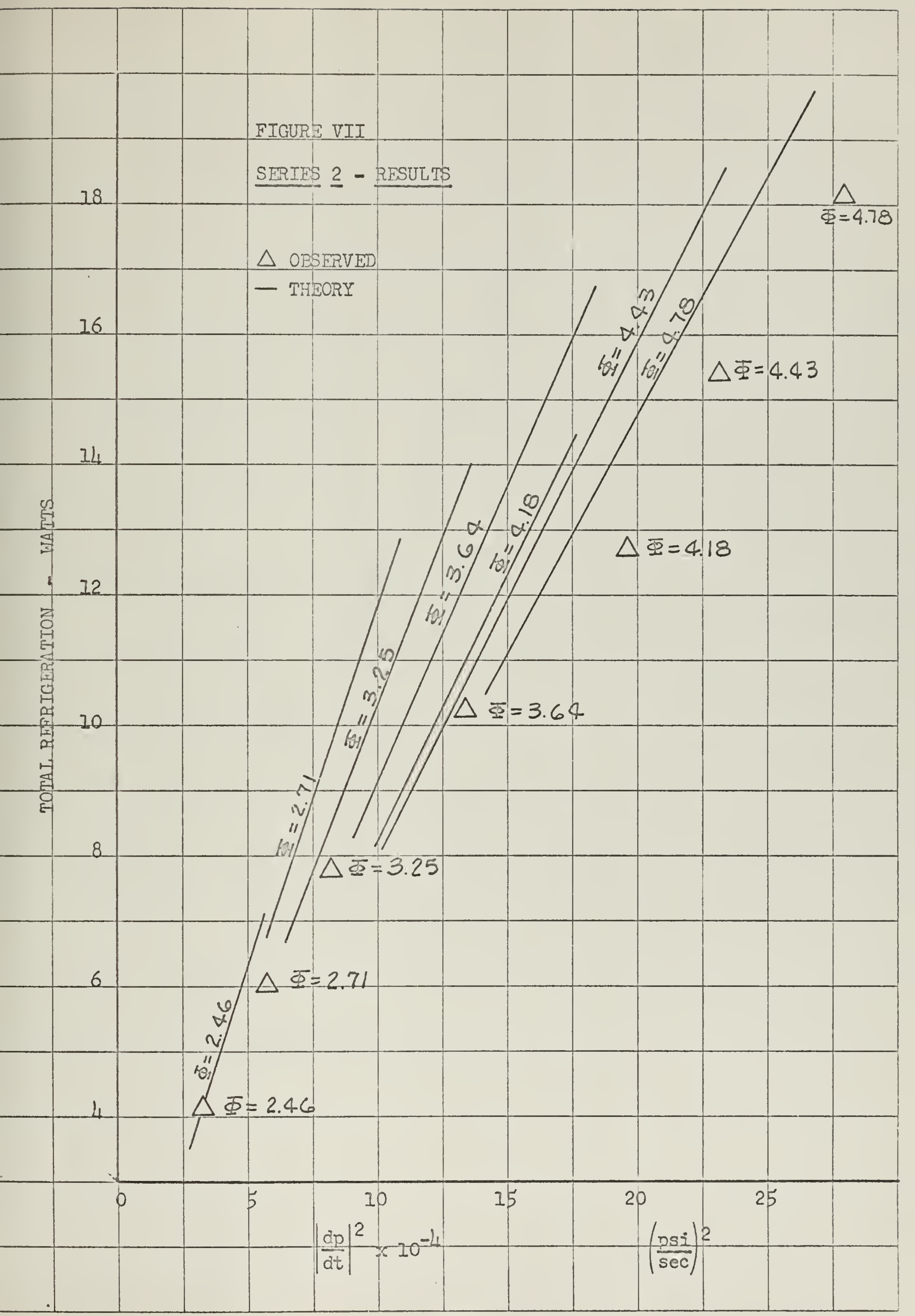


FIGURE VII
SERIES 2 - RESULTS
△ OBSERVED
— THEORY
TOTAL REFRIGERATION - WATTS
18
16
14
12
10
8
6
4
0
5
10
15
20
25
$\left|\frac{dp}{dt}\right|^2 \times 10^{-4}$
$\left(\frac{psi}{sec}\right)^2$
Φ = 2.46
Φ = 2.71
Φ = 3.25
Φ = 3.64
Φ = 4.18
Φ = 4.43
Φ = 4.78
△ Φ = 2.46
△ Φ = 2.71
△ Φ = 3.25
△ Φ = 3.64
△ Φ = 4.18
△ Φ = 4.43
△ Φ = 4.78

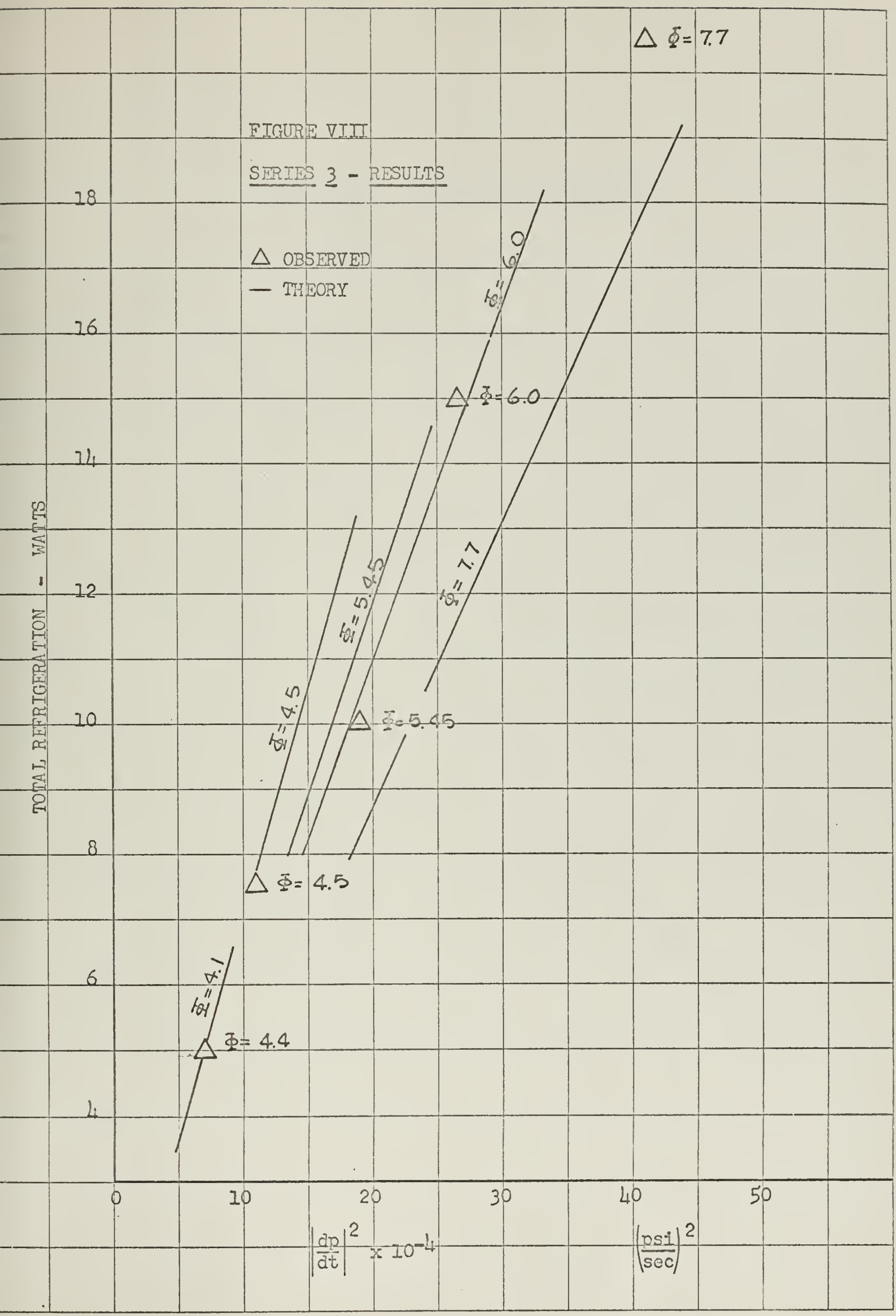


FIGURE VIII
SERIES 3 - RESULTS
△ OBSERVED
— THEORY
TOTAL REFRIGERATION - WATTS
18
16
14
12
10
8
6
4
0
10
20
30
40
50
$\left|\frac{dp}{dt}\right|^2 \times 10^{-4}$
$\left(\frac{psi}{sec}\right)^2$
Φ = 7.7
Φ = 6.0
Φ = 5.45
Φ = 4.5
Φ = 4.4
Φ = 4.1

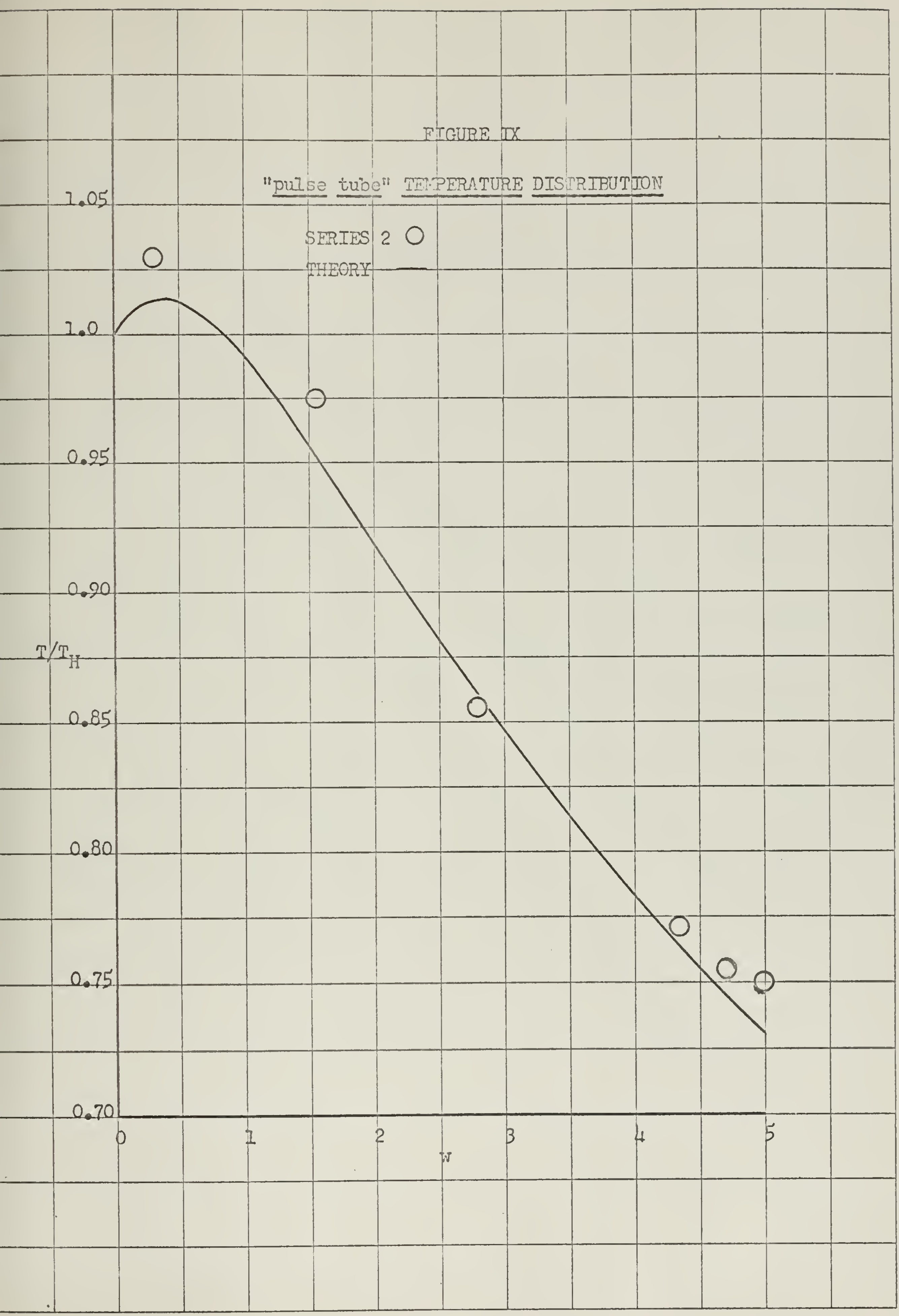


FIGURE IX
"pulse tube" TEMPERATURE DISTRIBUTION
SERIES 2
THEORY
1.05
1.0
0.95
0.90
T/T_H
0.85
0.80
0.75
0.70
0
1
2
3
4
5
w

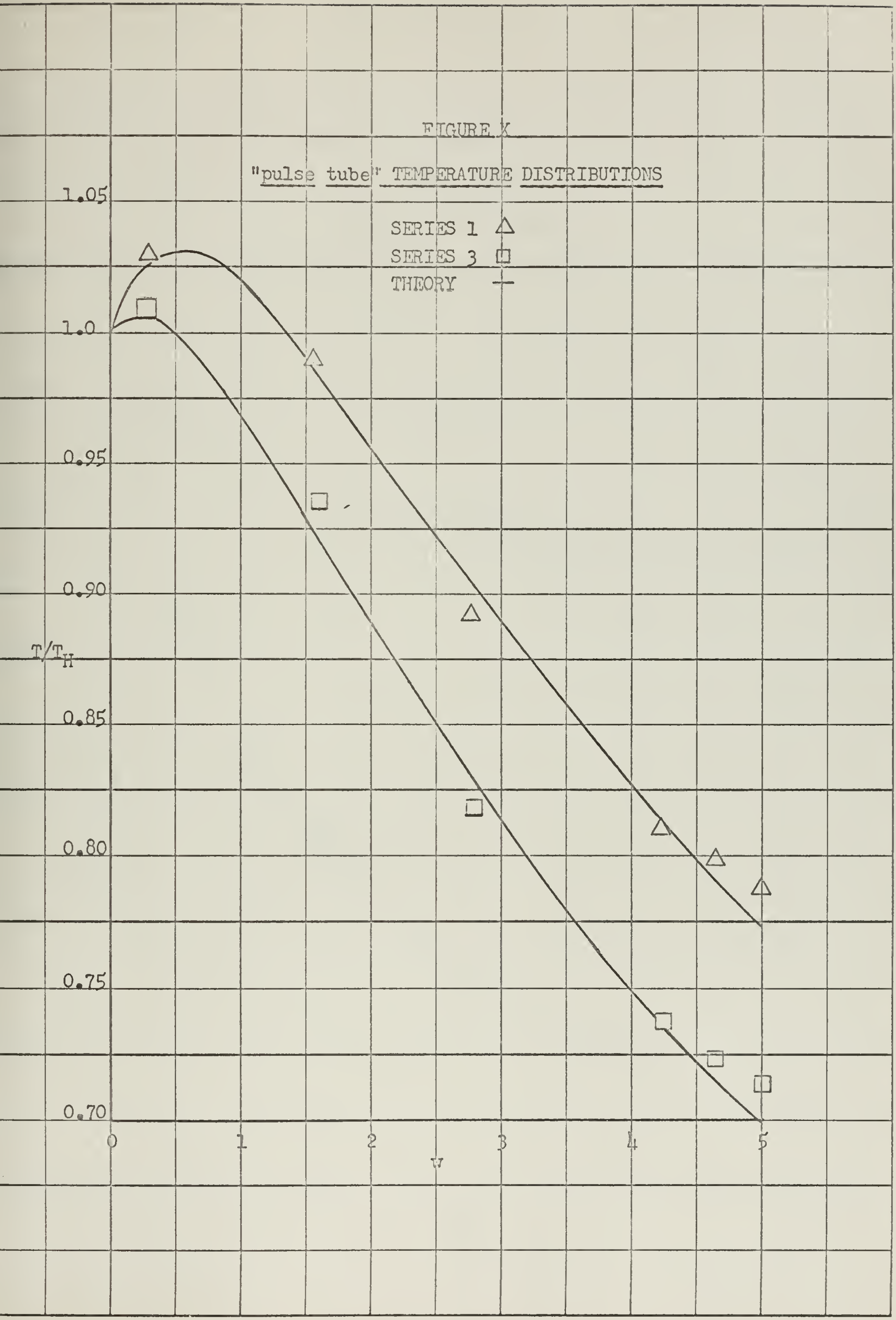


FIGURE X
"pulse tube" TEMPERATURE DISTRIBUTIONS
SERIES 1 △
SERIES 3 □
THEORY —
1.05
1.0
0.95
0.90
T/T_H
0.85
0.80
0.75
0.70
0
1
2
3
4
5
w

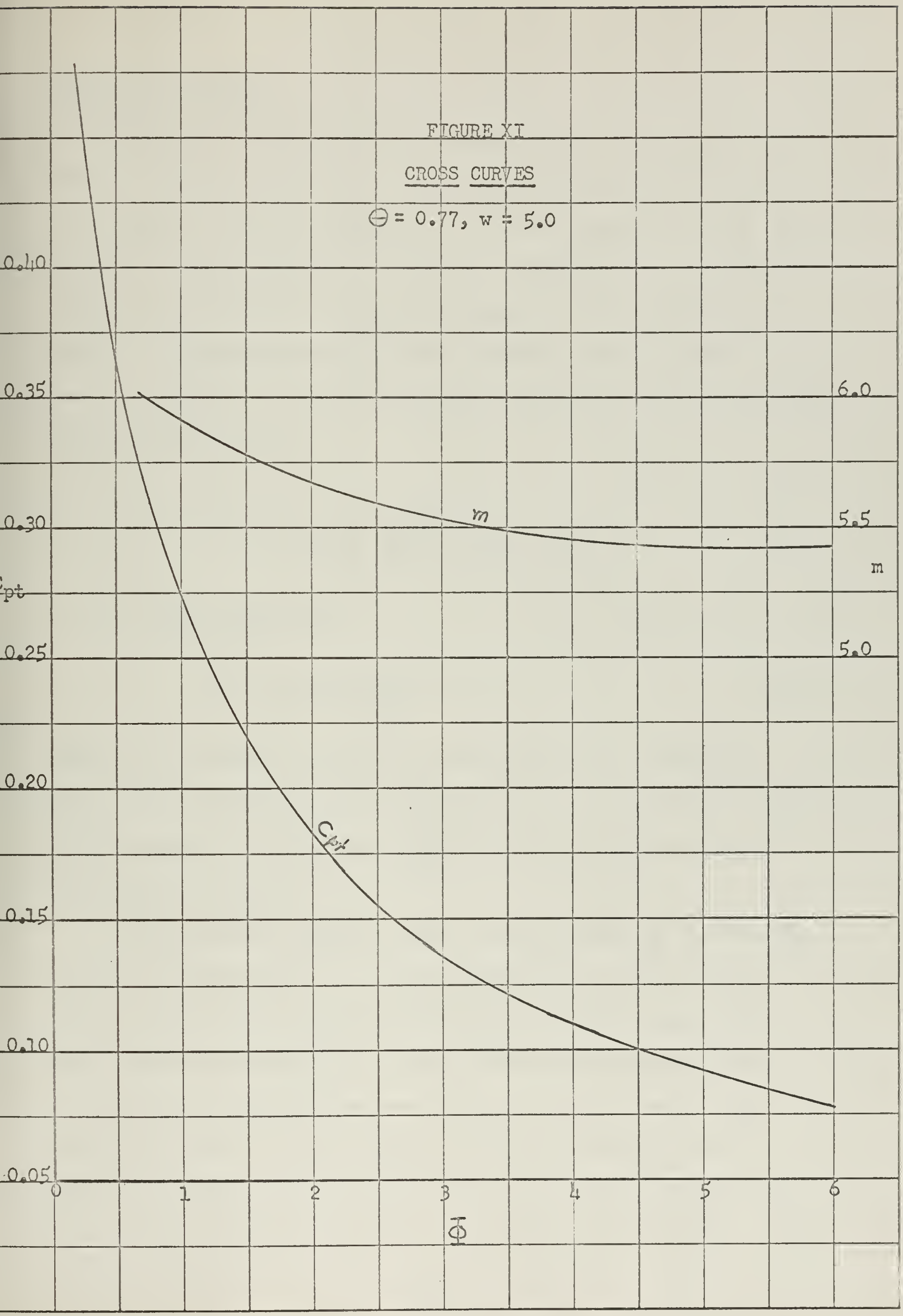


FIGURE XI
CROSS CURVES
Θ = 0.77, w = 5.0
0.40
0.35
0.30
C_{pt}
0.25
0.20
0.15
0.10
0.05
6.0
5.5
m
5.0
m
C_{pt}
0
1
2
3
4
5
6
Φ

Discussion of Results

From Tables I, II, and III it should be noted that αC_R is less than 10% of C_{pt} in all cases for this data and for practical purposes can be neglected when calculating $\dot{Q}_R$. The data indicates that as long as the regenerator used in the device is an efficient one, its physical dimensions relative to the pulse tube dimensions can be smaller than those of the regenerator used in this experiment before its effect would have to be considered. If the αC_R term of equation (3.17) is neglected, that equation becomes:

$$\dot{Q}_R = 2\psi\left(\frac{\gamma}{\gamma-1}\right)^2 \left|\frac{dp}{dt}\right|^2 \frac{V_H}{T_H}\left(\frac{\epsilon}{K\beta}\right)_{pt}\left(C_{pt}\right) \quad , \qquad (5.1)$$

or, for a given pulse tube,

$$\dot{Q}_R = \text{Constant}\left|\frac{dp}{dt}\right|^2 C_{pt} \quad , \qquad (5.2)$$

where C_{pt} is a function of Φ. All the Figures of $\dot{Q}_R$ versus $\left|\frac{dp}{dt}\right|^2$ in this chapter are plotted using equation (5.2). As a result, all lines of Φ = constant on the $\dot{Q}_R$ versus $\left|\frac{dp}{dt}\right|^2$ plots are straight lines passing through the origin.

For a particular series of runs a single temperature distribution was used in determining a C_{pt} for any specific run in that series, since all runs in each series had practically identical pulse tube axial temperature distributions. The observed and theoretical pulse tube temperature distributions are shown in Figures IX and X. The theoretical results for each series are plotted as single curves, since the deviation between runs in any one series was less than ½%. It is also interesting to note that the experimental curves are in good

agreement with the theoretical curves, indicating that the laminar flow assumption for the pulse tube is a good one.

From Figures VI through VIII one can determine approximately the accuracy of Rea's Modified Theory. However, the accuracy shown in these Figures is no better than the accuracy of the variables involved in arriving at these results. From equation (5.2) it can be seen that $\dot{Q}_R$ depends on $\left|\frac{dp}{dt}\right|^2$ and C_{pt}. The quantity C_{pt} is, in turn, dependent upon Φ. In determining $\left|\frac{dp}{dt}\right|$ and Φ, average values of pressure and time were used. The methods of determining these average values seem reasonable. However, other methods which would have resulted in different average values could also be justified. The importance of these results, then, lies not in attempting to show their accuracy, but in showing that the trend of the results can be explained by the theory.

The dimensionless parameters which evolve from the theory are most valuable in understanding the problem. For example, consider Runs 2 and 5 from Series 1 of Figure VI. Note that the $\left|\frac{dp}{dt}\right|^2$ is approximately the same for these two runs and yet $\dot{Q}_{R_{exper}}$ for Run 2 is more than twice $\dot{Q}_{R_{exper}}$ for Run 5. This can be explained, of course, by the fact that for Run 5 the value of Φ was almost twice as large as Φ for Run 2. As a result C_{pt} for Run 2 was more than one and one-half times as large as C_{pt} for Run 5. An indication of how C_{pt} varies with Φ in this case is shown in Figure XI. Thus, $\dot{Q}_{R_{theor}}$ for Run 5 was approximately one-half as large as $\dot{Q}_{R_{theor}}$ for Run 2. The large difference in Φ for these two runs is attributable to the fact that the back pressure for Run 5 was four times larger than the back pressure for Run 2; also Run 5 had a $\psi\tau$ of 0.27, while $\psi\tau$ for Run 2 was 0.50. The effect of these quantities on Φ can be easily seen by examining equation (2.35).

Another example of interest are Runs 5 and 4 of Series 1. Note that the total experimental refrigeration for these two runs is about equal, yet $\left|\frac{dp}{dt}\right|^2$ is more than twice as large for Run 5 as for Run 4. The explanation, of course, was that $\bar{\Phi}$ for Run 5 was almost twice as large as $\bar{\Phi}$ for Run 4. The values of $\bar{\Phi}$ were different, since $\psi\tau$ for Run 5 was 0.27 opposed to 0.50 for Run 4 and the back pressure for Run 5 was 20% greater for Run 5 than for Run 4. This resulted in C_{pt} for Run 4 being more than one and one-half times as large as C_{pt} for Run 5. This difference in C_{pt} due to $\bar{\Phi}$ was enough to counteract the opposite effect on $\dot{Q}_{R_{theor}}$ due to $\left|\frac{dp}{dt}\right|^2$ and, hence, both conditions resulted in producing approximately the same refrigeration effect.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

From the analytical model of Chapter II, there appeared three principal parameters which characterize the operation of a Pulse Tube. These parameters are:

$$C \equiv \frac{\gamma - 1}{\gamma}\left(\frac{KA_T L}{V}\right)\frac{C_1}{2\dot{m}_o}\frac{1}{\left|\frac{dp}{dt}\right|} , \tag{2.31}$$

$$w_{max} = \frac{V}{T_o \dot{m}_o R}\left|\frac{dp}{dt}\right| , \tag{2.32}$$

which were described by Rea (4), plus a third parameter $\bar{\Phi}$, defined as

$$\bar{\Phi} \equiv 2\left(\frac{\gamma}{\gamma - 1}\right)\left(\frac{V}{KA_T L}\right)\frac{p}{\Delta t}\frac{1}{T_o} . \tag{2.35}$$

The parameter $\bar{\Phi}$ resulted from the fact that the ratio $\left(\frac{1}{T_g}\frac{\partial p}{\partial t}\right) \Big/ \left(\frac{p}{T_g^2}\frac{\partial T_g}{\partial t}\right)$ for the pulse tube was much closer to unity than for the regenerator. Thus, the simplifying assumption which neglected the term $\left(\frac{p}{T_g^2}\frac{\partial T_g}{\partial t}\right)$ in the continuity equation (2.3) in the case of the regenerator could not be made in the case of the pulse tube.

From the theoretical and the experimental results obtained from the Pulse Tube under consideration, it was apparent that Rea's Modified Theory as presented in Chapter II does describe the performance of a

typical Pulse Tube. There was some deviation between the refrigeration effect measured and the refrigeration effect predicted. However, this deviation falls within acceptable limits for heat transfer data in the cyclic case.

It was apparent from the data obtained that there was good agreement between the experimental and the theoretical pulse tube axial temperature distributions. This leads one to the conclusion that laminar flow in the pulse tube is a good assumption.

From the analytical results obtained from the Pulse Tube, it was concluded that the effect of the regenerator on the overall refrigeration was negligible. Consequently, the regenerator could have been made smaller in this case before its effect would have become significant.

Experimental and theoretical results indicated that there is a unique temperature distribution for a pulse tube operating with a specified Θ_C.

Recommendations

There are several areas of Pulse Tube refrigeration which should be examined. First, the uniqueness of the axial temperature distributions of several pulse tubes with different values of w_{max} operating with a specific Θ_C should be investigated over a wide range of Φ. If this uniqueness can be demonstrated for a significant range of w_{max} and Φ, then a suggested method for presenting the computer solutions for the pulse tube differential equations is as follows:

(1) Plot for a fixed w_{max} a series of cross curves of C_{pt} versus Φ for various values of Θ_C.

(2) Introduce theoretical axial temperature distributions for

some value of Φ which covers the maximum feasible range in w. With the temperature distribution curves described above, the Θ_C and w_{max} of any pulse tube can be used to determine a curve of constant C_{pt}. By following along this constant C_{pt} curve to the value of w_{max} for which the cross curves of C_{pt} versus Φ apply, a corresponding value of Θ_C can be obtained. This value of Θ_C represents a single curve on the plot of the cross curves of C_{pt} and Φ. Thus, the relationship between C_{pt} and Φ becomes known.

Second, a method for determining an optimum size regenerator to be used with a given pulse tube should be devised.

Third, an optimizing procedure for the design of a pulse tube using Rea's Modified Theory should be developed.

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APPENDIX

APPENDIX A

REA'S 2-PART MODEL AND THE DEVELOPMENT OF THE GOVERNING DIFFERENTIAL EQUATIONS

General

As a mathematical model for his analysis of thermal regenerators subjected to rapid pressure and flow cycling, Rea devised his 2-Part Model. From this model Rea developed the differential equations that apply to regenerators subjected to the above conditions. Rea proceeded to apply certain assumptions which simplified these equations. The final result was a system of two differential equations, which Rea solved numerically. He then reasoned that since a pulse tube is simply an inefficient regenerator, the same differential equations should apply to it. However, not all of the assumptions which led to the regenerator differential equations hold true in the case of the pulse tube. Thus, the final differential equations which govern the pulse tube are not the same as the differential equations which govern the regenerator. This Appendix is broken into three sections. In Section A.1 Rea's 2-Part Model and his development of the differential equations for the regenerator will be presented in detail as in (4). Section A.2 deals with the development of the differential equations for the pulse tube. The latter section will not be presented in as rigorous a manner as Rea's development in Section A.1, since the two developments are similar. All assumptions made in each case will be described clearly. Section A.3 discusses the method of solution for the regenerator and pulse tube differential equations.

A.1

REA'S 2-PART MODEL AND DEVELOPMENT OF THE REGENERATOR DIFFERENTIAL EQUATIONS

Rea's 2-Part Model

Rea's 2-Part Model, which is shown in Figure II of Chapter I, is defined as a regenerator in which the mass flow rates at each end of the regenerator are in phase. By "in phase" it is meant that during each half-cycle the flow is always unidirectional within the regenerator, i.e. during the pressurization half-cycle, mass flows in at the top of regenerator and exits at the bottom, and during the expansion half-cycle, mass flows in at the bottom and exits at the top.

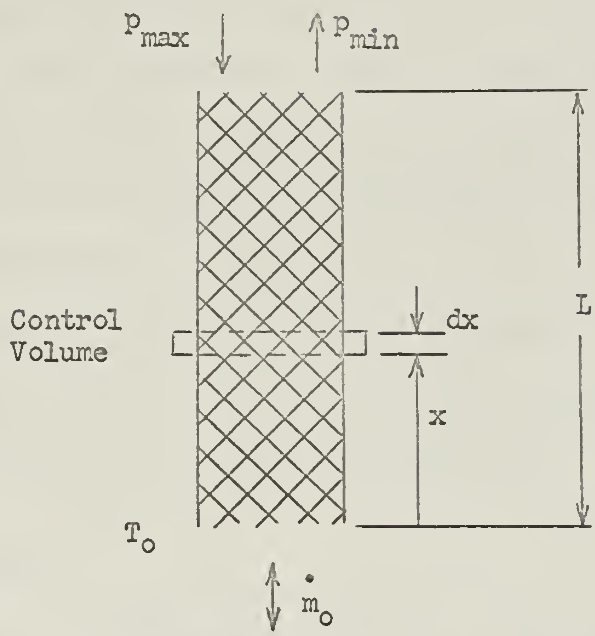
The operation of Rea's 2-Part Model is as follows: the inlet valve opens admitting high pressure gas into the regenerator. The pressure increases within the regenerator and the mass flow rate, $\dot{m}_o$, exits at the bottom. The mass flow rate at the top, $\dot{m}_W$, although unknown, is certainly greater than $\dot{m}_o$ since some gas must enter and occupy the volume within the regenerator. When some maximum pressure, p_{max}, is reached, the inlet valve closes and the exit valve opens. During the expansion half-cycle mass flow equal to $\dot{m}_o$ enters at the bottom and mass flow equal to $\dot{m}_W$ exits at the top. During this half-cycle the pressure decreases to its minimum value, p_{min}. Once a steady, cyclic state is reached, the above processes are repeated continuously with the temperature at the top equal to T_W and the temperature at the bottom equal to T_o.

It is interesting to note that, since $\dot{m}_W$ is always greater than $\dot{m}_o$, all the gas entering at the top does not reach the bottom. Thus,

it is possible for some gas to remain always within the regenerator and flow back and forth as the pressure cycles.

FIGURE XII

WORKING SCHEMATIC OF THE REGENERATOR



Development of the Governing Differential Equations for the Regenerator

The development which follows with regard to the regenerator differential equations is entirely attributable to Rea (4).

Rea's initial assumptions made concerning the development of the regenerator differential equations are as follows:

(1) The pressure drop in the regenerator is neglected,

(2) The gas obeys the perfect gas law,

(3) The gas transport properties and the matrix properties are constant,

(4) The cross sectional area of the regenerator open to flow, A_o, is constant.

Figure XII shows a schematic of the system to be analyzed. The system consists of a regenerator of length L, which is filled with a matrix material of large heat capacity, e.g. spherical lead shot. The subscript m is used to denote the matrix. The void volume within the regenerator is denoted by V_R and is equal to $A_o L$. During each cycle of operation the gas in the regenerator is alternately compressed and expanded between the pressures p_{max} and p_{min}.

In accordance with the assumptions listed previously, energy equations will be written for the gas and for the matrix contained within the control volume of length dx, as shown in Figure XII.

<u>Gas Energy Equation:</u>

Energy flux into the Control Volume = Energy flux out of the Control Volume + Rate of Change of Energy within the Control Volume

$$\dot{m}h + \left(- k_g A_o \frac{\partial T_g}{\partial x} \right) = \dot{m}h + \frac{\partial}{\partial x}\left(\dot{m}h \right) dx - \left[k_g A_o \frac{\partial T_g}{\partial x} + k_g A_o \frac{\partial}{\partial x}\left(\frac{\partial T_g}{\partial x} \right) dx \right]$$

$$+ h_T A_T \left(T_g - T \right) dx + \frac{\partial}{\partial t}\left(\rho_g u A_o dx \right) ,$$

or, upon simplifying the above equation,

$$\frac{\partial}{\partial x}\left(\dot{m}h \right) - k_g A_o \frac{\partial^2 T_g}{\partial x^2} + h_T A_T \left(T_g - T \right) + A_o \frac{\partial}{\partial t}\left(\rho_g u \right) = 0. \quad (A.1)$$

In equation (A.1), the first term is the change in enthalpy flux across the control volume, the second term represents the gradient of the energy transport by gas conduction, the third term is the convective heat transfer between the gas and the matrix, and the fourth term is the rate of change of energy within the control volume.

Matrix Energy Equation:

Energy flux into the Control Volume	=	Energy flux out of the Control Volume	+	Rate of change of Energy within the Control Volume.

$$-k_m A \frac{\partial T}{\partial x} + h_T A_T \left(T_g - T \right) dx = - \left[k_m A \frac{\partial T}{\partial x} + k_m A \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} \right) dx \right]$$

$$+ Mc_m \frac{\partial T}{\partial t} \; dx \quad ,$$

or, upon simplifying,

$$k_m A \frac{\partial^2 T}{\partial x^2} + h_T A_T \left(T_g - T \right) = Mc_m \frac{\partial T}{\partial t} \quad . \qquad (A.2)$$

In equation (A.2) the first term is the gradient of the energy transport by conduction along the matrix, the second term is the convective heat transfer from the gas to the matrix, and the term on the right is the rate of change of energy of the matrix.

Equations (A.1) and (A.2) are energy balances for the gas and matrix, respectively. Subscripts g and m refer to the gas and the matrix, respectively; h represents enthalpy and u represents internal energy; h_T is the heat transfer coefficient (which has not been assumed constant) and k is thermal conductivity; A is the total cross-sectional area of the regenerator and A_o is the cross-sectional area open to

flow ($A_o = \epsilon A$); A_T is the heat transfer surface area of the regenerator per unit length (ft^2/ft); and M is the matrix mass per unit length (lb_m/ft).

The gas continuity equation will now be developed.

Gas Continuity:

Mass flow into the Control Volume = Mass flow out of the Control Volume + Rate of storage of mass within the Control Volume

$$\dot{m} = \left(\dot{m} + \frac{\partial \dot{m}}{\partial x} dx \right) + \frac{\partial}{\partial t}\left(\rho_g A_o dx \right) ,$$

or,

$$\frac{\partial \dot{m}}{\partial x} = - A_o \frac{\partial \rho_g}{\partial t} . \qquad (A.3)$$

Equation (A.3) is the continuity equation and it is perfectly general (within the assumptions listed previously), as it applies to either steady or unsteady flow. With the use of the perfect gas law,

$$\rho_g = \frac{p}{RT_g} , \qquad (A.4)$$

equation (A.3) becomes

$$\frac{\partial \dot{m}}{\partial x} = - \frac{A_o}{R} \frac{\partial}{\partial t}\left(\frac{p}{T_g} \right) . \qquad (A.5)$$

Equations (A.1), (A.2), and (A.3) along with appropriate boundary conditions are the governing differential equations for any type of regenerator operation. Equation (A.1) will now be simplified by using equations (A.4) and (A.5).

For a perfect gas, one may write

$$h = h_o + c_p T_g \tag{A.6}$$

and

$$u = h - pv = h - RT_g = h_o + c_v T_g \quad ,$$

where h_o is the gas enthalpy at some reference temperature.

The first term in equation (A.1) becomes, with the use of equations (A.5) and (A.6):

$$\frac{\partial}{\partial x}\left(\dot{m}h\right) = h_o \frac{\partial \dot{m}}{\partial x} + c_p \frac{\partial}{\partial x}\left(\dot{m}T_g\right) = -\frac{A_o h_o}{R} \frac{\partial}{\partial t}\left(\frac{p}{T_g}\right) + c_p \frac{\partial}{\partial x}\left(\dot{m}T_g\right) . \tag{A.7}$$

The fourth term in equation (A.1) becomes, with the use of equations (A.4) and (A.6):

$$A_o \frac{\partial}{\partial t}\left(\rho_g u\right) = \frac{A_o h_o}{R} \frac{\partial}{\partial t}\left(\frac{p}{T_g}\right) + \frac{c_v A_o}{R} \frac{\partial p}{\partial t} . \tag{A.8}$$

The substitution of equations (A.7) and (A.8) into equation (A.1) yields:

$$c_p \frac{\partial}{\partial x}\left(\dot{m}T_g\right) - k_g A_o \frac{\partial^2 T_g}{\partial x^2} + h_T A_T \left(T_g - T\right) + \frac{c_v A_o}{R} \frac{\partial p}{\partial t} = 0 . \tag{A.9}$$

Equation (A.9) is the final form of the energy equation for the gas.

Equations (A.9), (A.2), and (A.5), along with the appropriate boundary conditions, completely describe the performance of thermal regenerators consistent with the assumptions listed on page 57.

The assumption will now be made that terms involving thermal

conductivity are negligible in comparison to the other terms in (A.9) and (A.2). That this assumption is consistent with some practical cases is shown by taking some typical values:

$k_g = 0.07$ BTU/hr-ft-^oF, $A_o = 0.00194$ ft^2, $(T_g - T) = 2^oF$,

$A_T = 4.43$ ft^2/ft, $h_T = 235$ BTU/hr-ft^2-^oF, $\frac{\partial^2 T_g}{\partial x^2} \simeq \frac{T_W - T_o}{L^2} = 150^oF/ft^2$.

Then:

$$k_g A_o \frac{\partial^2 T_g}{\partial x^2} \doteq 0.0204 \quad \text{BTU/hr-ft, and}$$

$$h_T A_T (T_g - T) = 2080 \quad \text{BTU/hr-ft.}$$

Thus, the gas conduction term is approximately 0.001% of the convective heat transfer term of equation (A.9).

In the matrix energy equation, (A.2), with $k_m \simeq 20$ BTU/hr-ft-^oF and $A = 0.005$ ft^2:

$$k_m A \frac{\partial^2 T}{\partial x^2} \simeq 15 \quad \text{BTU/hr-ft.}$$

Therefore the conduction term for the matrix is less than 1% of the convective heat transfer term. Then neglecting axial conduction, the final basic differential equations become:

$$c_p \frac{\partial}{\partial x}\left(\dot{m} T_g\right) + h_T A_T \left(T_g - T\right) + \frac{c_v A_o}{R} \frac{\partial p}{\partial t} = 0 \tag{A.10}$$

$$h_T A_T \left(T_g - T\right) = M c_m \frac{\partial T}{\partial t} \tag{A.11}$$

$$\frac{\partial \dot{m}}{\partial x} = -\frac{A_o}{R}\frac{\partial}{\partial t}\left(\frac{p}{T_g}\right) \quad . \tag{A.5}$$

In equation (A.10), the first term is the gradient of the enthalpy flux along the regenerator; the second term is the energy flux due to convective heat transfer; the third term is the work of compression. Equation (A.11) means that the increase in internal energy of the matrix equals the energy given up by the gas by convective heat transfer. Equation (A.5) is the gas continuity equation.

Boundary conditions for the above differential equations are:

$$\begin{aligned} \text{at } x = 0: \quad & \dot{m} = \dot{m}_o(t) \\ & T_g = T_{go}(t) \\ \text{at } x = L: \quad & T_g = T_{gN}(t) \end{aligned} \tag{A.12}$$

Simplification of the Differential Equations

Equations (A.10), (A.11), and (A.5) are three equations in three unknowns: T, T_g, and $\dot{m}$. Thus, if p as a function of t is known, then solutions consistent with the boundary conditions, equations (A.12), will exist. Unfortunately, equations (A.10), (A.11), and (A.5) are a system of differential equations for which no general solution is known. The method of approach, then, will be to determine solutions which satisfy certain conditions over a complete cycle, but which are not exact in every detail at each instant within the cycle.

A cyclic integral of equation (A.11), while holding x constant, yields:

$$\oint hA_T \left(T_g - T\right) \; dt = \oint Mc_m \frac{\partial T}{\partial t} dt = 0. \tag{A.13}$$

The above follows from the fact that once steady, cyclic conditions have been established, the matrix temperature at a given point in the regenerator oscillates around a value of the temperature which is a function of x only. From this point on, h will be used for the heat transfer coefficient rather than h_T.

A cyclic integral of equation (A.10) with x constant yields:

$$c_p \oint \frac{\partial}{\partial x}\left(\dot{m}T_g\right) dt + \oint hA_T \left(T_g - T\right) \; dt + \frac{c_v A_o}{R} \oint \frac{\partial p}{\partial t} \; dt = 0. \tag{A.14}$$

But, $\oint \frac{\partial p}{\partial t} \; dt = 0$ by definition, since p is a function of t only (no pressure drop in the regenerator). Then by using equation (A.13), equation (A.14) becomes:

$$\oint \frac{\partial}{\partial x}\left(\dot{m}T_g\right) dt = 0. \tag{A.15}$$

Since $\dot{m}T_g$ will be continuous in x, the order of differentiation and integration can be interchanged and equation (A.15) becomes:

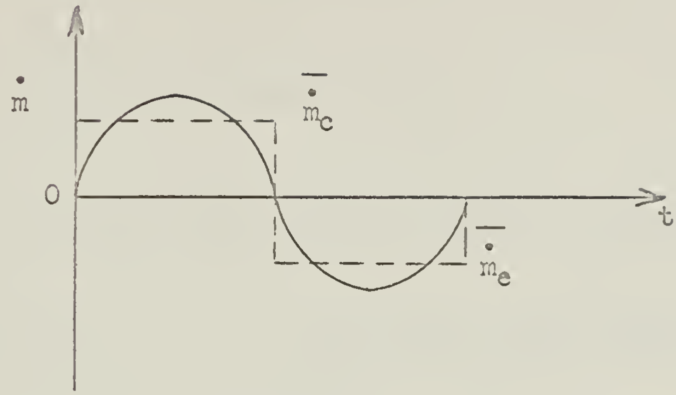
$$\frac{\partial}{\partial x} \oint \left(\dot{m}T_g\right) dt = 0 \quad ,$$

or,

$$\oint \left(\dot{m}T_g\right) dt = C_o \quad \text{(for all x)}. \tag{A.16}$$

For the 2-Part Model, the pressurization part of the cycle is denoted by the subscript c and the expansion part of the cycle by the subscript e. The mass flow rate at a given point within the regenerator will be cyclic, passing through zero at reversal. The mass flow versus

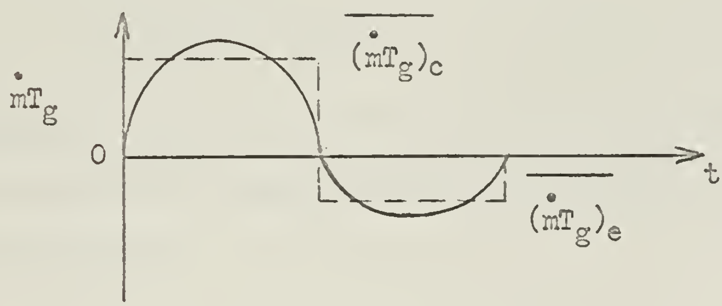
time might appear as:



Since there can be no net mass storage within the regenerator once steady, cyclic conditions prevail,

$$\oint \dot{m}dt = 0 \tag{A.17}$$

and the positive area must exactly cancel the negative area in the above diagram. Similarly, T_g will be cyclic with time, but in some unspecified manner and $\oint T_g\, dt$ will not, in general, equal zero. Then, the product $\dot{m}T_g$ might have a shape as follows:



where the $\oint \dot{m}T_g\, dt = C_o$ from equation (A.16). This means that the positive and negative areas in the above sketch do not, in general, cancel each other. The assumption will now be made that $\dot{m}T_g$ versus time can be represented by the dashed lines in the above figure.

Then:

$$\oint \dot{m}T_g dt = \int_c \dot{m}_c T_{gc} dt_c + \int_e \dot{m}_e T_{ge} dt_e = C_o$$

$$\simeq \overline{\dot{m}_c T_{gc}} \tau_c + \overline{\dot{m}_e T_{ge}} \tau_e = C_o \quad , \qquad (A.18)$$

where τ_c is the blow time during compression and τ_e is the blow time during expansion.

Furthermore, if $\tau_c = \tau_e$ (compression period equals expansion period), equation (A.18) can be written:

$$\overline{\dot{m}_c T_{gc}} + \overline{\dot{m}_e T_{ge}} = C_1 \quad , \qquad (A.19)$$

where

$$C_1 = C_o/\tau_c = C_o/\tau_e \quad .$$

From equation (A.17),

$$\oint \dot{m} dt = \int_c \dot{m}_c dt + \int_e \dot{m}_e dt = 0 \quad ,$$

and,

$$\int_c \dot{m}_c dt = \int_e \dot{m}_e dt \quad . \qquad (A.20)$$

With the assumption that the mass flow rate can be represented by an average flow rate which is constant with time over each half-cycle, equation (A.20) becomes:

$$\overline{\dot{m}_c} \tau_c = - \overline{\dot{m}_e} \tau_e \quad ,$$

and again letting $\tau_c = \tau_e$,

$$\overline{\dot{m}_c} = \overline{\dot{m}_e} \quad . \tag{A.21}$$

Equation (A.20) and, hence, equation (A.21) holds at any given x along the regenerator. From here on, from the result of equation (A.21), $\overline{\dot{m}}$ will be used without subscript to denote flow in either the compression or expansion half-cycle and $\overline{\dot{m}}$ is a function of x only.

With the assumption that

$$\overline{\dot{m}T} = (\overline{\dot{m}})(\overline{T}) \quad ,$$

equation (A.19) along with (A.21) becomes

$$\overline{\dot{m}}\,(\overline{T_{gc}} - \overline{T_{ge}}) = C_1 \tag{A.22}$$

Equation (A.22) is a very important and useful result. There are no restrictions on C_1. It can be positive, negative, or zero, depending upon whether $\overline{T_{gc}}$ is greater than, less than, or equal to $\overline{T_{ge}}$.

The usual heat transfer correlation is assumed,

$$h = K_1 \dot{m}^n \quad ,$$

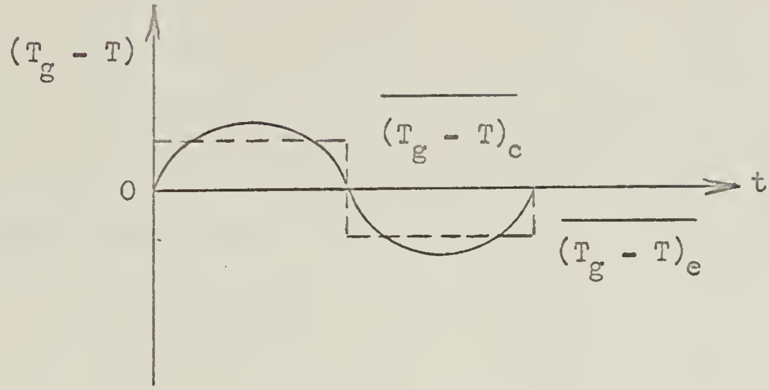
where n is a positive exponent whose value depends on the particular regenerator matrix. In general $0.5 \leq n \leq 0.7$. Hence, h is either constant or a function of $\dot{m}$ only. Using equation (A.21), h will be a function of x only and will be written as h. Then from equation (A.13):

$$\oint \overline{h}\, A_T\, (T_g - T)\, dt = \overline{h}\, A_T \oint (T_g - T)\, dt = 0 \quad ,$$

since $\overline{h}$ is independent of t, or,

$$\int_c (T_g - T)\, dt = - \int_e (T_g - T)\, dt \quad . \tag{A.23}$$

A plot of $(T_g - T)$ versus t might appear as:



From equation (A.23), the positive area on the above sketch must equal the negative area. As before, assuming $(T_g - T)$ can be represented by some average value and letting $\tau_c = \tau_e$, equation (A.23) becomes:

$$\overline{(T_g - T)_c} = \overline{(T_g - T)_e} \tag{A.24}$$

Since the blow-time, τ, is usually short, a reasonable assumption to make is that the temperature of the matrix at a given x will vary very little during a half-cycle and, in fact, T can be assumed constant with time. In that case, $T = T(x)$. The validity of this assumption can be demonstrated by the following example. From equation (A.11):

$$h A_T (T_g - T) = Mc_m \frac{\partial T}{\partial t} \tag{A.11}$$

For a regenerator 12" long and 1" diameter packed with lead shot of 0.050" diameter, $Mc_m L = .068$ BTU/°F. Taking $h = 235$ BTU/hr-ft^2-°F, $A_T L = 4.43$ ft^2, and assuming $(T_g - T)$ 2°F:

$$hA_T (T_g - T) = 2080 \text{ BTU/hr} = 0.068 \frac{\partial T}{\partial t} \quad ,$$

and $$\frac{\partial T}{\partial t} = 30{,}600 \ {}^{o}F/hr = 8.5 \ {}^{o}F/sec \quad .$$

With a blow time of 0.1 second, $\Delta T = 0.85 \ {}^{o}F$. During a complete cycle, at a given x, the matrix temperature would oscillate around a mean temperature with an amplitude on the order of 0.85 °F. Thus, the assumption is made that T is independent of time and is a function of x only.

By applying the above assumption to equation (A.24), one is lead to the important result that:

$$\overline{T_{gc}} + \overline{T_{ge}} = 2\overline{T} \tag{A.25}$$

At first glance equation (A.25) seems obvious, but a careful examination of the preceeding discussion will show that two key assumptions were necessary in deriving equation (A.25). The assumptions were (1) the mass flow rate, $\dot{m}$, is a function of x only, and (2) T is a function of x only.

By carrying out the time-differentiation in the continuity equation (A.5), one obtains:

$$\frac{\partial \dot{m}}{\partial x} = -\frac{A_o}{R}\left(\frac{1}{T_g}\frac{\partial p}{\partial t} - \frac{p}{T_g^{\,2}}\frac{\partial T_g}{\partial t}\right) \quad . \tag{A.26}$$

The ratio of the first term in the brackets of equation (A.26) to the second term is:

$$\frac{1}{T_g}\frac{\partial p}{\partial t} \Big/ \frac{p}{T_g^{\,2}}\frac{\partial T_g}{\partial t} \simeq \frac{\Delta p}{p}\frac{T_g}{\Delta T_g} \simeq \frac{T_g}{\Delta T_g} \quad .$$

With $\Delta T_g \simeq \Delta T = 0.85 \ {}^{o}F$, the above ratio becomes:

$$\frac{T_g}{\Delta T_g} \simeq 1.2T_g = 430 \text{ to } 625.$$

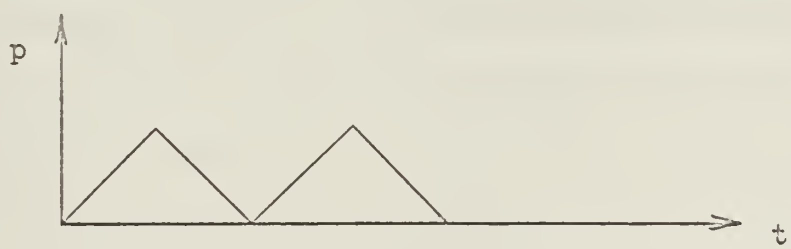
Thus, the second term in the brackets of equation (A.26) is less than 1% of the first term and, consequently, the second term will be neglected. Equation (A.26) can now be written as:

$$\frac{d\overline{\dot{m}}}{dx} = \frac{A_o}{RT_g} \frac{dp}{dt} \quad . \tag{A.27}$$

Total derivatives are used in equations (A.27) since p is a function of t only and $\overline{\dot{m}}$ is considered to be a function of x only. Furthermore, T_g will be replaced by T in equation (A.27) since (T_g - T) is small (2° F to 5° F). This assumption also enables continuity to be satisfied over a cycle. Since $\overline{\dot{m}}$ has already been assumed to be equal and opposite for the compression and expansion half-cycles, it is reasonable to expect $\frac{d\dot{m}}{dx}$ to be the same for both half-cycles. Then the continuity equation is written as:

$$\frac{d\overline{\dot{m}}}{dx} = - \frac{A_o}{RT} \frac{dp}{dt} \tag{A.28}$$

It is now assumed that $\frac{dp}{dt} = \pm$ constant. Thus, pressure is assumed to be a sawtooth function of time as shown in the sketch below, where the zero-axis is any arbitrary pressure level:



Letting $\frac{dp}{dt} = \pm$ constant also maintains consistency with the previous assumptions that $\dot{m}$ and T are functions of x only.

It is now possible to present the final formulation of the regenerator problem. From Figure XII, the positive x direction for the regenerator is from bottom to top. Mass flow in this direction is positive. During pressurization the flow is assumed to be from top to bottom, or, in the negative x direction. The gas energy equation which was derived for positive $\dot{m}$ with positive $\frac{dp}{dt}$, is written for the two half-cycles:

$$-c_p \frac{d}{dx}\left(\dot{m}T_{gc}\right) + hA_T\left(T_{gc} - T\right) + \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad . \qquad (A.29)$$

$$c_p \frac{d}{dx}\left(\dot{m}T_{ge}\right) + hA_T\left(T_{ge} - T\right) - \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad . \qquad (A.30)$$

In equations (A.29) and (A.30), $\left|\frac{dp}{dt}\right|$ is written as an absolute value with the plus or minus sign placed in front, depending upon whether pressurization or expansion is occurring in the regenerator. The continuity relation, equation (A.28), is written:

$$\frac{d\dot{m}}{dx} = \frac{A_o}{RT}\left|\frac{dp}{dt}\right| \quad . \qquad (A.31)$$

For convenience, average value bars have been omitted. Equation (A.31) holds for both half-cycles, since in the compression half-cycle $\dot{m}$ is negative, but $\frac{dp}{dt}$ is positive, whereas in the expansion half-cycle $\dot{m}$ is positive, but $\frac{dp}{dt}$ is negative.

The final formulation of the problem consists, then, of equations

(A.29), (A.30), and (A.31) along with the results previously attained:

$$\overset{\circ}{m}\left(T_{gc} - T_{ge}\right) = C_1 \tag{A.22}$$

$$T_{gc} + T_{ge} = 2T \quad . \tag{A.25}$$

Boundary conditions for the above set of equations are:

$$\text{at } x = 0: \quad T = T_o$$

$$\overset{\circ}{m} = \overset{\circ}{m}_o$$

$$\text{at } x = L: \quad T = T_W \quad . \tag{A.32}$$

Equations (A.29), (A.30), (A.31), (A.22), and (A.25) are five equations in five unknowns: $\overset{\bullet}{m}$, T, T_{gc}, T_{ge}, and C_1. However, there are only three boundary conditions. Therefore, it is necessary to eliminate the gas temperatures from the above equations.

If the differentiation indicated in equations (A.29) and (A.30) is carried out there results:

$$-c_p\left(\overset{\bullet}{m}\frac{dT_{gc}}{dx} + T_{gc}\frac{d\overset{\bullet}{m}}{dx}\right) + hA_T\left(T_{gc} - T\right) + \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad . \tag{A.29a}$$

$$c_p\left(\overset{\bullet}{m}\frac{dT_{ge}}{dx} + T_{ge}\frac{d\overset{\circ}{m}}{dx}\right) + hA_T\left(T_{ge} - T\right) - \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad . \tag{A.30a}$$

Now if equation (A.30a) is subtracted from equation (A.29a) the result is:

$$-c_p\left[\overset{\bullet}{m}\left(\frac{dT_{gc}}{dx} + \frac{dT_{ge}}{dx}\right) + \left(T_{gc} + T_{ge}\right)\frac{d\overset{\circ}{m}}{dx}\right] + hA_T\left(T_{gc} - T_{ge}\right)$$
$$+ \frac{2c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad . \tag{A.33}$$

Now by substituting equations (A.22), (A.25), and (A.31), plus the relationship $R = c_p - c_v$, equation (A.33) reduces to:

$$\dot{m}\,\frac{dT}{dx} - \frac{hA_T C_1}{2c_p\dot{m}} + \frac{A_o}{c_p}\left|\frac{dp}{dt}\right| = 0 \qquad , \qquad (A.34)$$

which involves only the matrix temperature, T.

The following new variables are now introduced:

$$\begin{aligned} \Theta &= T/T_o \\ m &= \dot{m}/\dot{m}_o \\ w &= \frac{V}{T_o\dot{m}_o R}\left|\frac{dp}{dt}\right|\frac{x}{L} \end{aligned} \qquad . \qquad (A.35)$$

Also the heat transfer coefficient is now assumed to be correlated by

$$h = Km^n \qquad , \qquad (A.36)$$

where K is a constant with the units $BTU/hr\text{-}ft^2\text{-}{}^oF$.

With the aid of the new variables, equation (A.36), and the definition $\gamma = \dfrac{c_p}{c_v}$, equation (A.34) can be written:

$$m^{2-n}\,\frac{d\Theta}{dw} + \frac{\gamma-1}{\gamma}\,m^{1-n} = C \qquad (A.37)$$

where

$$C = \frac{\gamma-1}{\gamma}\,\frac{KA_T L C_1}{2\dot{m}_o V\left|\frac{dp}{dt}\right|} \qquad . \qquad (A.38)$$

The continuity relation, equation (A.31), becomes

$$\frac{dm}{dw} = \frac{1}{\Theta} \qquad (A.39)$$

Boundary conditions for equations (A.37) and (A.39) are:

$$\text{at } w = 0: \quad m = 1$$

$$\Theta = 1$$

$$\text{at } w = w_{max} = \frac{V}{T_o \dot{m}_o R} \left| \frac{dp}{dt} \right| : \Theta = \Theta_W \qquad (A.40)$$

Equations (A.37) and (A.39), along with the boundary conditions (A.40), are sufficient to determine m, T, and the constant C. Actually, since everything in the definition of C is known except C_1, determining C is essentially determining C_1.

Thus the problem for the regenerator is reduced to one of solving a set of two ordinary differential equations subject to the boundary conditions mentioned. The method of solution will be discussed later in section A.3.

A.2

THE GOVERNING DIFFERENTIAL EQUATIONS FOR REA'S 2-PART MODEL AS APPLIED TO A "pulse tube"

A pulse tube, as long as its heat transfer area per unit total volume is fairly large, can be considered to be an inefficient regenerator. Thus, a similar development to that leading to the final differential equations for the regenerator could be presented for the pulse tube. However, such a development would be repetitious. The procedure that will be followed will be to accept the assumptions listed on page 57 and start with equations (A.9), (A.5), and (A.2), which are the gas energy equation, the gas continuity equation, and the matrix energy equation respectively. Appropriate assumptions regarding the relative magnitude of the axial conductivity will then be applied. Next the cyclic conditions will be presented. At this point the relative magnitude of the terms of the continuity equation will be examined. The final formulation of the pulse tube problem will then be possible. The pulse tube differential equations will evolve in much the same form as equations (A.37) and (A.39). It will be of some assistance to present here a schematic for the 2-Part Model for a pulse tube. This is shown in Figure XIII. Note that x is measured positive from the top down, which is directly opposite from the case of the regenerator.

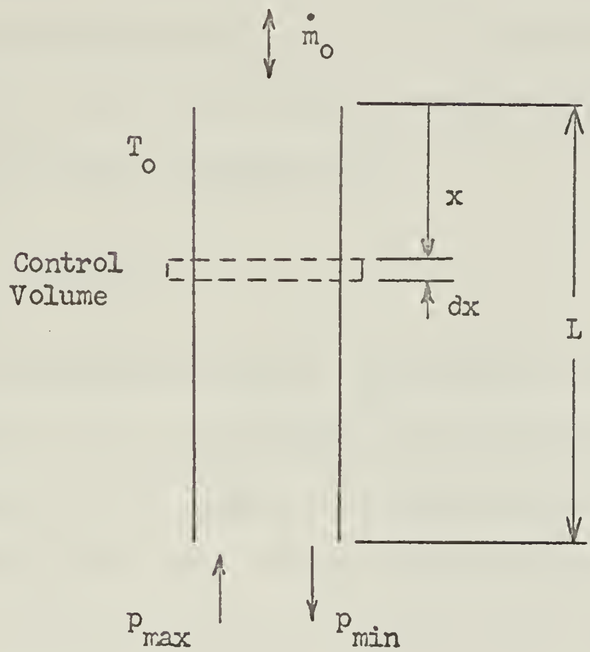
Equations (A.9), (A.5), and (A.2) in the form in which they were previously developed are as follows:

$$c_p \frac{\partial}{\partial x}\left(\dot{m}T_g\right) - k_g A_o \frac{\partial^2 T_g}{\partial x^2} + hA_T\left(T_g - T\right) + \frac{c_v A_o}{R}\frac{\partial p}{\partial t} = 0 \qquad (A.9)$$

$$\frac{\partial \dot{m}}{\partial x} = -\frac{A_o}{R}\frac{\partial}{\partial t}\left(\frac{p}{T_g}\right) \qquad (A.5)$$

FIGURE XIII

SCHEMATIC OF THE SYSTEM TO BE ANALYZED FOR THE "pulse tube"



$$k_m A \frac{\partial^2 T}{\partial x^2} + h_T A_T\left(T_g - T\right) = Mc_m \frac{\partial T}{\partial t} \quad . \qquad (A.2)$$

As in the case of the regenerator, the assumption will now be made that the terms involving thermal conductivity are negligible in comparison with the other terms in equations (A.9) and (A.2). Some typical values which substantiate this assumption are as follows:

$k_g = 0.07$ BTU/hr-ft-°F, $A_o = 0.0019$ ft^2, $(T_g - T) = 2$°F,

$A_T = 0.155\ ft^2/ft, \qquad h_T = 6.18\ BTU/hr\text{-}ft^2\text{-}{}^oF,$

$$\frac{\partial^2 T_g}{\partial x^2} \simeq \frac{T_H - T_o}{L^2} = 120^oF/ft^2.$$

Then:

$$k_g A_o \frac{\partial^2 T_g}{\partial x^2} = 0.016 \quad BTU/hr\text{-}ft, \quad \text{and}$$

$$h_T A_T\, (T_g - T) = 1.92 \quad BTU/hr\text{-}ft.$$

Thus, the gas conduction term is less than 1% of the convective heat transfer term of equation (A.9). In the matrix energy equation (A.2), with $k_m \simeq 10$ BTU/hr-ft-^oF and $A = 0.000225\ ft^2$:

$$k_m A \frac{\partial^2 T}{\partial x^2} \simeq 0.270 \quad BTU/hr\text{-}ft$$

the conduction term is approximately 14% of the convective term. If the device were operated with a $(T_H - T_o)$ of 190^{o}F, this percentage would increase to approximately 22%. Obviously this assumption is not as good as in the regenerator case, but without it the problem would become exceedingly complex. Therefore equations (A.9) and (A.2) become respectively:

$$c_p \frac{\partial}{\partial x}\left(\dot{m} T_g\right) + hA_T \left(T_g - T\right) + \frac{c_v A_o}{R} \frac{\partial p}{\partial t} = 0 \tag{A.10}$$

and

$$h_T A_T \left(T_g - T\right) = Mc_m \frac{\partial T}{\partial t} \quad . \tag{A.11}$$

The cyclic conditions for the pulse tube are identical with those for the regenerator. Thus, equation (A.22) applies equally as well to the pulse tube as to the regenerator. Again the heat transfer

correlation is assumed to be:

$$h = K_1 \dot{m}^n \quad .$$

For the pulse tube, however, laminar flow is assumed. At the moment it is not intuitively obvious that this is a valid assumption. However, experimental data indicates that laminar flow is probably the closest approximation that can be made for the pulse tube. Thus, $n = 0$, and the heat transfer correlation for the pulse tube becomes:

$$h = \text{constant} \quad .$$

Now, proceeding with an identical argument as that used in the case of the regenerator, one obtains equation (A.24):

$$\overline{(T_g - T)}_c = - \overline{(T_g - T)}_e \tag{A.24}$$

Here the assumption is made that the temperature of the pulse tube wall at a given value of x will vary little during a half-cycle. Therefore T can be assumed to be constant with time, and $T = T(x)$ only. The validity of this assumption can be shown by the following example. From equation (A.11):

$$hA_T(T_g - T) = Mc_m \frac{\partial T}{\partial t} \tag{A.11}$$

For a stainless steel pulse tube 12" long with O.D. = 0.625" and I.D. = 0.591", $Mc_mL = 0.011$ BTU/°F, and taking $h = 6.18$ BTU/hr-ft²-°F, $A_TL = 0.155$ ft², and assuming $(T_g - T)$ 2°F:

$$hA_T(T_g - T) = 1.92 \quad \text{BTU/hr} = 0.011 \ \frac{\partial T}{\partial t} \quad ,$$

and

$$\frac{\partial T}{\partial t} = 174.5 \ ^{\circ}\text{F/hr.} \ = \ 0.0485 \ ^{\circ}\text{F/sec.}$$

Thus, with a blow-time of 0.1 second, $\Delta T \simeq 0.005^{o}F$. During a complete cycle, at a given x, the pulse tube wall temperature would oscillate around a mean temperature with an amplitude of approximately $0.005^{o}F$.

Thus, the assumption that T is independent of time and is only a function of x, is a valid one.

By applying this assumption to equation (A.24), one obtains, once again, the result:

$$\overline{T_{gc}} + \overline{T_{ge}} = 2\overline{T} \quad . \tag{A.25}$$

By carrying out the time differentiation of the continuity equation (A.5), one obtains:

$$\frac{\partial \dot{m}}{\partial x} = - \frac{A_o}{R}\left(\frac{1}{T_g}\frac{\partial p}{\partial t} - \frac{p}{T_g^2}\frac{\partial T_g}{\partial t}\right) \quad . \tag{A.26}$$

The ratio of the first term in the brackets of equation (A.26) to the second term is:

$$\frac{1}{T_g}\frac{\partial p}{\partial t} \Big/ \frac{p}{T_g^2}\frac{\partial T_g}{\partial t} \simeq \frac{\Delta p}{p}\frac{T_g}{\Delta T_g} \quad .$$

From equation (A.22):

$$\Delta T_g = \frac{C_1}{\dot{m}} \tag{A.41}$$

From equation (A.38):

$$C_1 = 2C \frac{\gamma}{\gamma - 1} \dot{m}_o \frac{V}{KA_T L} \left| \frac{dp}{dt} \right| \quad . \tag{A.42}$$

By substituting equation (A.42) into equation (A.41), one obtains:

$$\Delta T_g = 2C \frac{\gamma}{\gamma-1} \frac{\dot{m}_o}{\dot{m}} \frac{V}{KA_TL} \left| \frac{dp}{dt} \right| \quad . \tag{A.43}$$

Thus,

$$\frac{\Delta p}{p} \frac{T_g}{\Delta T_g} = \frac{1}{2C} \frac{\gamma-1}{\gamma} \frac{\dot{m}}{\dot{m}_o} \frac{KA_TL}{V} \frac{\Delta t}{p} T_g \quad .$$

Now using typical values of $\gamma = 1.67$, $KA_TL/V = 41.6$ BTU/in-hr-ft^2-^{o}F, $C = .2$, $\dot{m}/\dot{m}_o = 3$, $\Delta t = .27$ sec, $p = 75$ psia, $T_g = 390^oF$, the above ratio becomes:

$$\frac{\Delta p}{p} \frac{T_g}{\Delta T_g} \simeq 3$$

For the regenerator this same ratio was 430 to 625, and the second term was neglected. Rea's theory also neglects this term for the pulse tube. However, the above example indicates that the second term cannot be neglected in the case of the pulse tube. Equation (A.26), then, remains as:

$$\frac{d\overline{\dot{m}}}{dx} = - \frac{A_o}{R} \left(\frac{1}{T_g} \frac{dp}{dt} - \frac{p}{T_g^{\,2}} \frac{dT_g}{dt} \right) \quad . \tag{A.44}$$

Total derivatives are used because p is assumed to be a function of t only, T_g (at a given x) is considered to be a function of t only, and $\overline{\dot{m}}$ is considered to be a function of x only. Also, since $(T_g - T)$ is small ($2^oF.$ to $5^oF.$), T_g will be replaced by T in equation (A.43). Then the continuity equation can be written as:

$$\frac{d\dot{m}}{dx} = - \frac{A_o}{R} \left(\frac{1}{T} \frac{dp}{dt} - \frac{p}{T^2} \frac{dT_g}{dt} \right) \quad . \tag{A.45}$$

Now the assumption is made that $\frac{dT_g}{dt}$ can be approximated by $\frac{\Delta T_g}{\Delta t}$. Once again $\frac{dp}{dt}$ is assumed to be constant for both half-cycles. Now by substituting equation (A.43) into equation (A.45) and using the fact that $\frac{dp}{dt} = \pm$ constant, equation (A.45) becomes:

$$\frac{d\dot{m}}{dx} = \frac{A_o}{R}\left[\frac{1}{T}\left|\frac{dp}{dt}\right| - 2C\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\Delta t}\frac{1}{T^2}\left(\frac{\dot{m}_o}{\dot{m}}\right)\left(\frac{V}{KA_T L}\right)\left|\frac{dp}{dt}\right|\right] \qquad (A.46)$$

or, simplifying,

$$\frac{d\dot{m}}{dx} = \frac{A_o}{RT}\left|\frac{dp}{dt}\right|\left(1 - B\right) \qquad (A.47)$$

where

$$B \equiv 2C\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\Delta t}\frac{1}{T}\left(\frac{\dot{m}_o}{\dot{m}}\right)\left(\frac{V}{KA_T L}\right) \quad . \qquad (A.48)$$

In equations (A.46) and (A.47) average value bars have been omitted for convenience.

As in the case of the regenerator, the gas energy equation can be written for the two half-cycles:

$$-c_p\frac{d}{dx}\left(\dot{m}T_{gc}\right) + hA_T\left(T_{gc} - T\right) + \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad . \qquad (A.29)$$

$$c_p\frac{d}{dx}\left(\dot{m}T_{ge}\right) + hA_T\left(T_{ge} - T\right) - \frac{c_v A_o}{R}\left|\frac{dp}{dt}\right| = 0 \quad . \qquad (A.30)$$

The final formulation of the pulse tube problem consists, then, of equations (A.47), (A.29), (A.30) along with the results:

$$\dot{m}\left(T_{gc} - T_{ge}\right) = C_1 \quad , \tag{A.22}$$

$$T_{gc} + T_{ge} = 2T \quad . \tag{A.25}$$

Boundary conditions for the above set of equations are, as for the regenerator:

$$\text{at } x = 0: \quad T = T_o \tag{A.49}$$

$$\dot{m} = \dot{m}_o$$

$$\text{at } x = L: \quad T = T_C$$

Note that the condition at $x = L$ is different for the pulse tube than for the regenerator only in the fact that the <u>cold</u> end of the pulse tube is at $x = L$ whereas the <u>warm</u> end of the regenerator was at $x = L$. Thus the subscript C stands for "cold" in the pulse tube boundary condition, whereas the subscript W stood for "warm" in the regenerator boundary condition.

As in the case for the regenerator, equations (A.29), (A.30), (A.47), (A.22), and (A.25) are five equations in five unknowns: $\dot{m}$, T, T_{gc}, T_{ge}, and C_1. Since there are only three boundary conditions, it is necessary to eliminate the gas temperatures from the above equations.

If equation (A.30) is subtracted from equation (A.29) and if equations (A.22), (A.25), and (A.47) are used to eliminate the gas temperatures, the resulting equation involves only the temperature of the pulse tube wall:

$$\dot{m}\frac{dT}{dx} - \frac{hA_T C_1}{2c_p\dot{m}} + \frac{A_o}{c_p}\left|\frac{dp}{dt}\right|\left[\frac{\left|1-B\right|\gamma -1}{\gamma -1}\right] = 0 \quad . \tag{A.50}$$

Now if B is replaced by equation (A.48), equation (A.50) becomes:

$$\dot{m}\,\frac{dT}{dx} - \frac{hA_TC_1}{2c_p\dot{m}} + \frac{A_o}{c_p}\left|\frac{dp}{dt}\right|\left[\frac{\left(1 - 2C\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\Delta t}\,\frac{1}{T}\left(\frac{\dot{m}_o}{\dot{m}}\right)\left(\frac{V}{KA_TL}\right)\right)\gamma-1}{\gamma-1}\right] = 0\ . \qquad (A.51)$$

The variables Θ, m, and w, equation (A.35), plus the heat transfer correlation, $h = K$, are now introduced into equation (A.51), obtaining:

$$m^2\,\frac{d\Theta}{dw} + m\left(\frac{\gamma-1}{\gamma}\right) = C\left(1 + \frac{\Phi}{\Theta}\right)\quad , \qquad (A.52)$$

where

$$\Phi \equiv 2\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{V}{KA_TL}\right)\frac{p}{\Delta t}\,\frac{1}{T_o}\quad , \qquad (A.53)$$

Δt is Δt_{avg} for the entire cycle, and p is p_{avg} for the cycle which is taken as:

$$p_{avg} = p_{min} + \frac{p_{max} - p_{min}}{2} \qquad (A.54)$$

The continuity equation (A.47) becomes in this case:

$$\frac{dm}{dw} = \frac{1}{\Theta}\left(1 - \frac{\Phi C}{\Theta m}\right)\ . \qquad (A.55)$$

The boundary conditions for equations (A.52) and (A.54) are:

$$\text{at } w = 0: \quad m = 1, \quad \Theta = 1 \qquad (A.56)$$

$$\text{at } w = w_{max} = \frac{V}{T_o\dot{m}_oR}\left|\frac{dp}{dt}\right| : \Theta = \Theta_C\ .$$

Thus, equations (A.52) and (A.55) together with the boundary conditions (A.56) describe the problem for the pulse tube.

METHOD OF SOLUTION FOR THE DIFFERENTIAL EQUATIONS

Regenerator

If equation (A.37) is solved for $\frac{d\Theta}{dw}$,

$$\frac{d\Theta}{dw} = \frac{C - \frac{\gamma - 1}{\gamma} m^{1-n}}{m^{2-n}} \tag{A.57}$$

Also,

$$\frac{dm}{dw} = \frac{1}{\Theta} \quad . \tag{A.39}$$

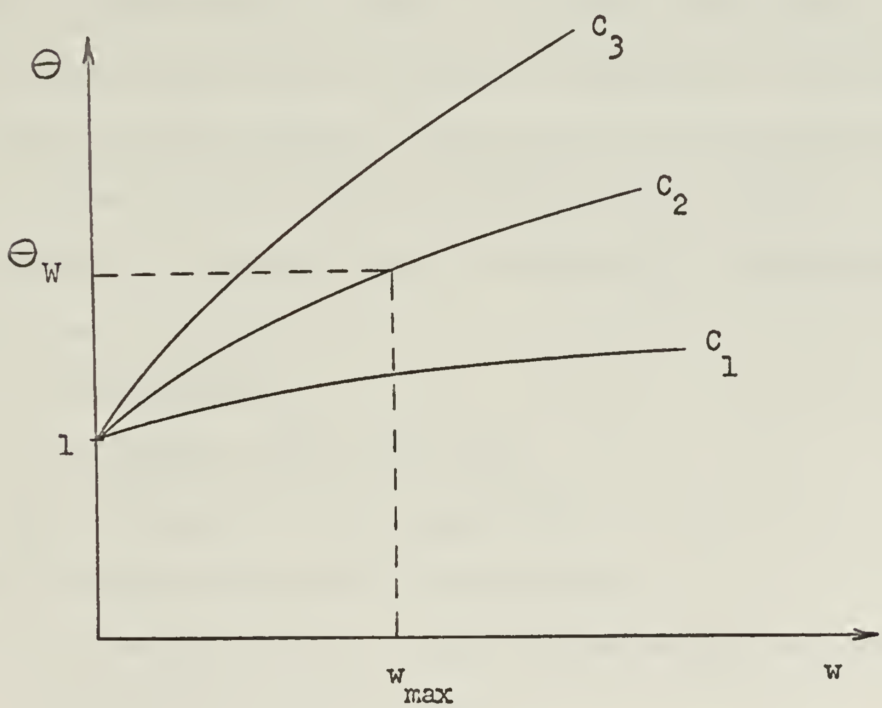
The simplest method of solution for the set of equations (A.57) and (A.39) is to solve them numerically with computer aid. The simplest approach is to treat C as a parameter and assign a range of values to it. The problem then becomes an initial value problem, since everything on the right side of equations (A.57) and (A.39) at $w = 0$ is known. The independent variable w can then be run out to sufficiently large values to include any desired w_{max}. Θ_W then becomes an unknown quantity, but can be determined from a plot of Θ versus w for a given value of C. On the other hand, for a given value of Θ_W and w_{max}, the corresponding value of C can be read from the same plot.

Figure XIV shows a hypothetical set of solutions corresponding to a family of C's. From the figure, for $C = C_2$, $\Theta = \Theta_W$ at $w = w_{max}$. The entire regenerator temperature distribution is determined by following the curve for $C = C_2$ from $w = 0$ to $w = w_{max}$. In like manner one can generate curves of m versus w from equation

(A.39). In this case the mass flow distribution can be determined by following the appropriate C curve from $w = 0$ to $w = w_{max}$. In addition to specifying a range of C's, the regenerator matrix must be known as well as the working gas, so that values of n and γ can be assigned in equation (A.57).

FIGURE XIV

ILLUSTRATION OF COMPUTER SOLUTIONS FOR REGENERATOR



"pulse tube"

If equation (A.52) is solved for $\frac{d\Theta}{dw}$,

$$\frac{d\Theta}{dw} = \frac{C\left(1 + \frac{\bar{\Phi}}{\Theta}\right) - m\left(\frac{\gamma - 1}{\gamma}\right)}{m^2} \qquad (A.58)$$

Also

$$\frac{dm}{dw} = \frac{1}{\Theta}\left(1 - \frac{\bar{\Phi} C}{\Theta m}\right) \quad . \qquad (A.55)$$

Again, the simplest method of solution of these equations is by computer. Similar to the case for the regenerator, the simplest approach is to treat $\bar{\Phi}$ and C as parameters. Thus, for a given $\bar{\Phi} = \bar{\Phi}_1$, C is assigned a range of values; then a new value is chosen for $\bar{\Phi}$ and C is assigned another range of values. The result is a family of curves of C for a given value of $\bar{\Phi}$, as is shown in Figure XV. The curves are used in exactly the same manner as that described for the regenerator. To use the curves one must be able to determine $\bar{\Phi}$ in advance. From equation (A.53) it is apparent that to determine $\bar{\Phi}$, the following information must be known:

(1) the working gas,

(2) the geometry of the pulse tube,

(3) the cycling rate (See Chapter IV),

(4) the temperature of the cooling medium,

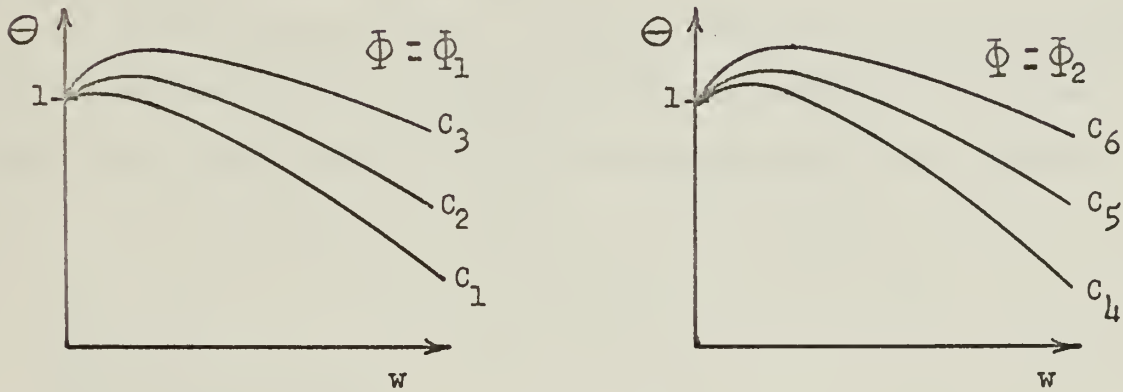
(5) the average pressure at which the device will be operated (defined by equation (A.54)).

Details of the computer program for the regenerator equations can be found in (4). Details of the computer program for the pulse tube

equations can be found in Appendix B. Computer solutions for the regenerator and pulse tube can be found in Appendix C.

FIGURE XV

ILLUSTRATION OF COMPUTER SOLUTIONS FOR "pulse tube"



APPENDIX B

COMPUTER PROGRAM FOR THE SOLUTION OF THE "pulse tube" EQUATIONS

General

Rea (4) wrote a program in Fortran to solve the regenerator equations (2.29) and (2.30). A similar program was written to solve the pulse tube equations (2.33) and (2.34). Runge's recurrence formula (6) was used in the stepwise integration.

The program was written for $\gamma = 1.67$. FUNF(B,R) is equation (2.33), where B corresponds to the mass flow, m, and R corresponds to the dimensionless temperature ratio, Θ . The other symbols used in the program are:

PAR = C

SPAR = Φ

TEMP = Θ = T/T_o

Y = m = $\dot{m}/\dot{m}_o$

SLOPE = $d\Theta/dw$

D = m^* in Runge recurrence formula

E = Θ^* in Runge recurrence formula

H = integration step size = 0.01

K = number of integration steps

KK = number of values of C for each value of Φ

KKK = number of values of Φ

The first data card contains the initial values of Θ and m, the step size, the number of integration steps, the number of values of C for each value of Φ , and the number of values of Φ . The remaining

data cards contain the values of Φ and C for which results are desired.

Computer Program

```
*       XEQ
*       LABEL
*       LIST8
*       FORTRAN
        FUNF(B,R)=(PAR*(1.+SPAR/R)-(0.4*B))/(B*B)
        DIMENSION A(   ),C(   ),TEMP(   ),Y(   ),SLOPE(   ),W(   )
        READ 10,TEMP(1),Y(1),H,K,KK,KKK
        DO 25 L=1,KKK
        READ 12,A(L)
        SPAR=A(L)
 3      DO 25 I=1,KK
        READ 12,C(I)
        PAR=C(I)
 6      DO 20 J=1,K
        D=Y(J)+(H*(1./TEMP(J))*(1.-((SPAR*PAR)/(TEMP(J)*Y(J))))/2.)
        E=TEMP(J)+(H*FUNF(Y(J),TEMP(J))/2.)
        TEMP(J+1)=TEMP(J)+(H*FUNF(D,E))
        Y(J+1)= Y(J)+(H*(1./E)*(1.-((SPAR*PAR)/(E*D))))
        W(J+1)=H*FLOATF(J)
 20     SLOPE(J+1)=FUNF(Y(J+1),TEMP(J+1))
        PRINT 13,H,PAR,SPAR
 25     PRINT 15,(W(J+1),TEMP(J+1),Y(J+1),SLOPE(J+1),J=20,K,20)
        CALL EXIT
 10     FORMAT (3F12.6,I4,I2,I2)
 12     FORMAT (1F12.6)
 13     FORMAT (3F12.6)
 15     FORMAT (4F15.5)
        END
*       DATA
 1.0         1.0         .01         600 710
 (OTHER DATA FOLLOWS)
```

APPENDIX C

THEORETICAL AXIAL TEMPERATURE

AND AXIAL MASS FLOW DISTRIBUTIONS,

2-PART MODEL.

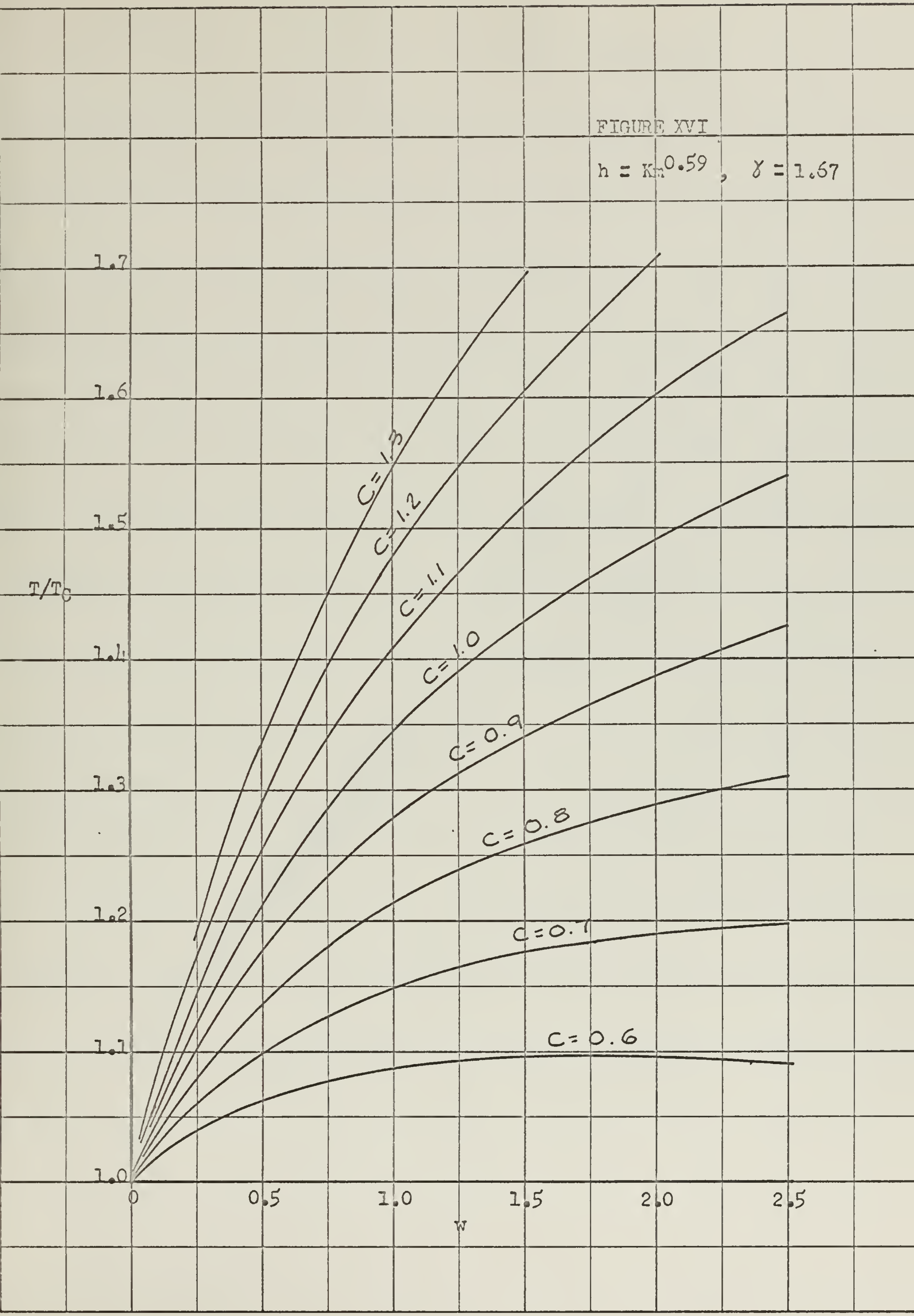
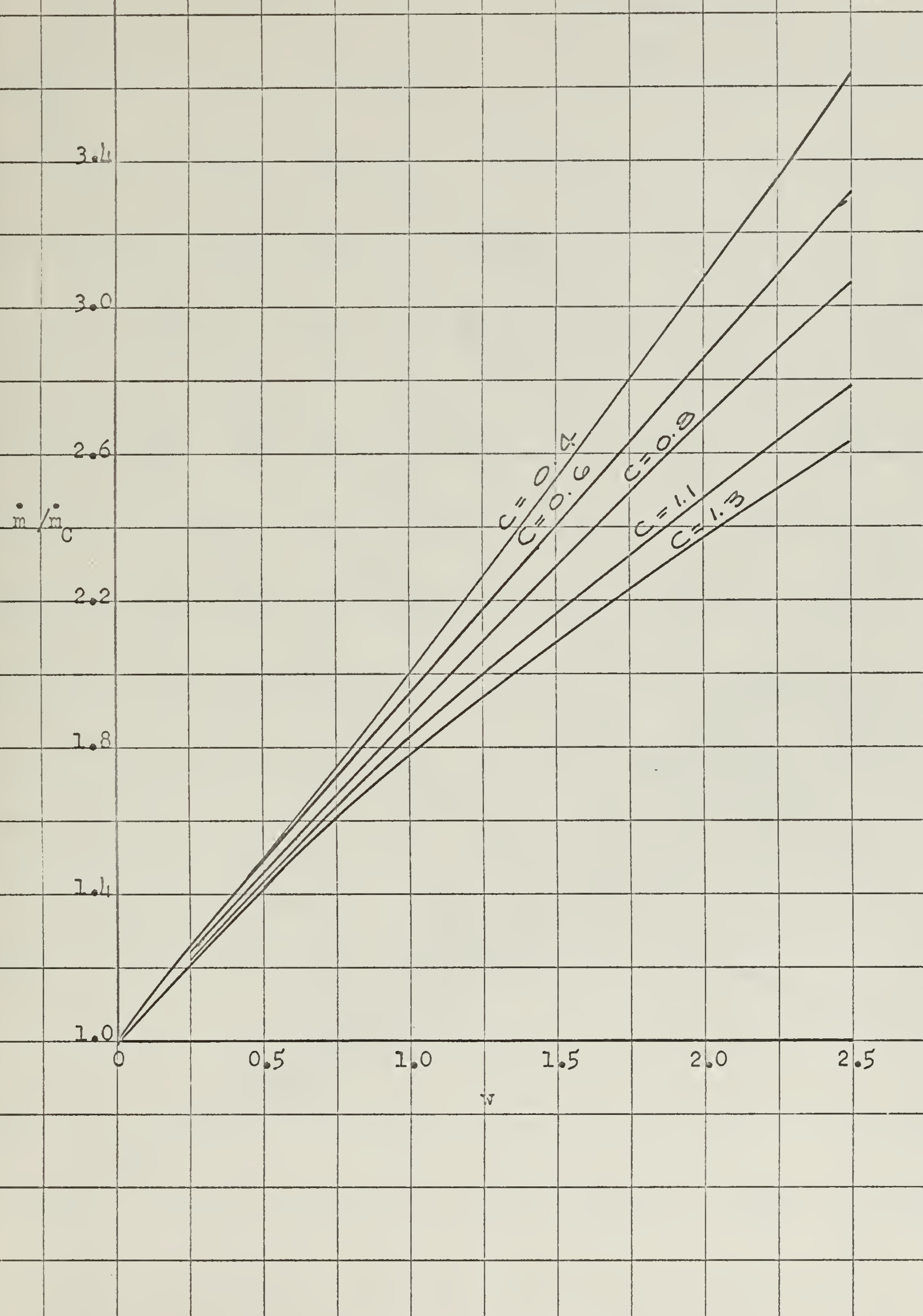


FIGURE XVI
h = Km0.59 , γ = 1.67
T/TC
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0
0.5
1.0
1.5
2.0
2.5
W
C = 1.3
C = 1.2
C = 1.1
C = 1.0
C = 0.9
C = 0.8
C = 0.7
C = 0.6

FIGURE XVII

$h = Km^{0.59}$, $\gamma = 1.67$



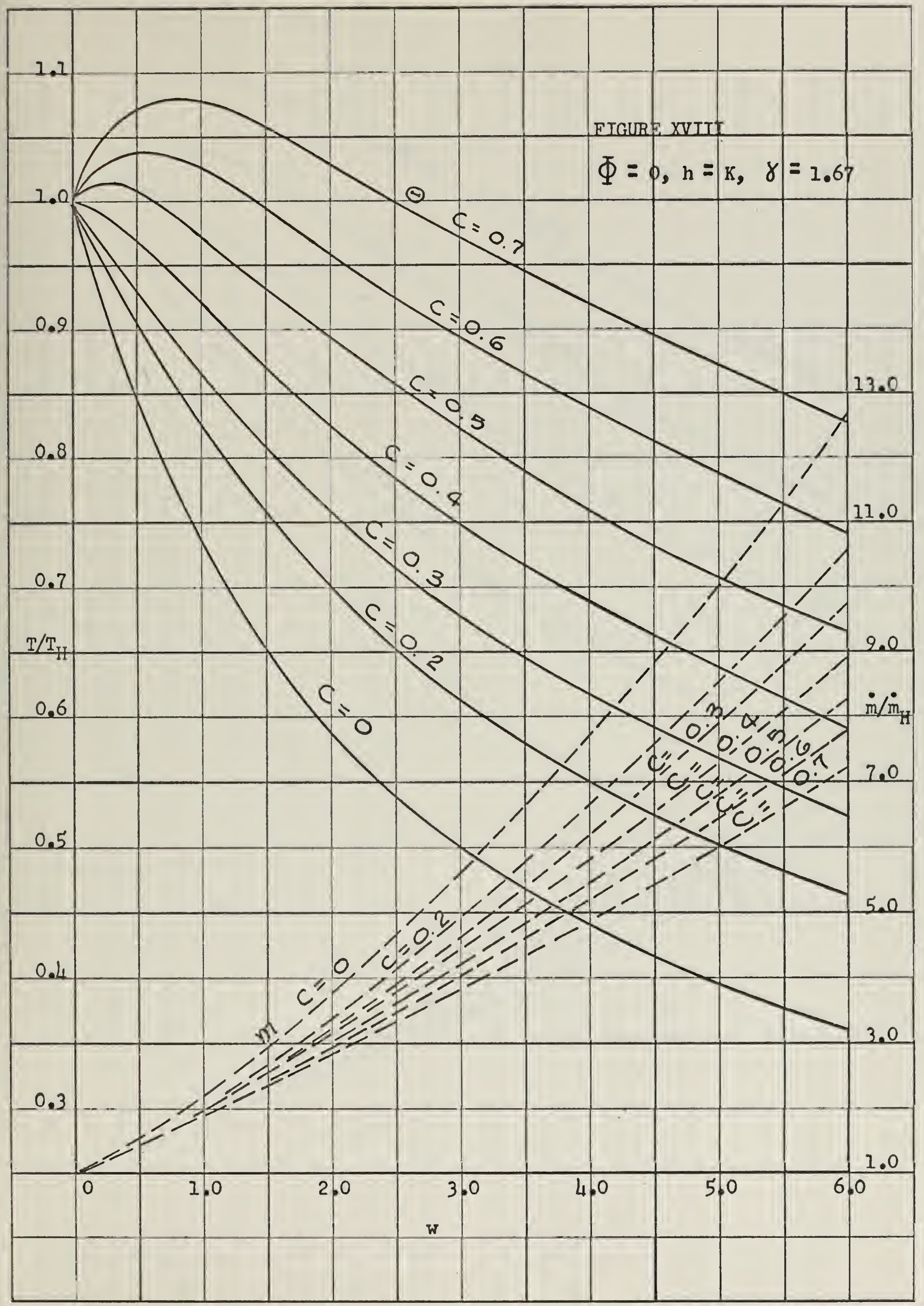


FIGURE XVIII
$\Phi = 0$, $h = K$, $\gamma = 1.67$
T/T_H
$\dot{m}/\dot{m}_H$
w
C = 0.7
C = 0.6
C = 0.5
C = 0.4
C = 0.3
C = 0.2
C = 0
Θ
$\dot{m}$
C = 0
C = 0.2
C = 0.3
C = 0.4
C = 0.5
C = 0.6
C = 0.7
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
13.0
11.0
9.0
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0

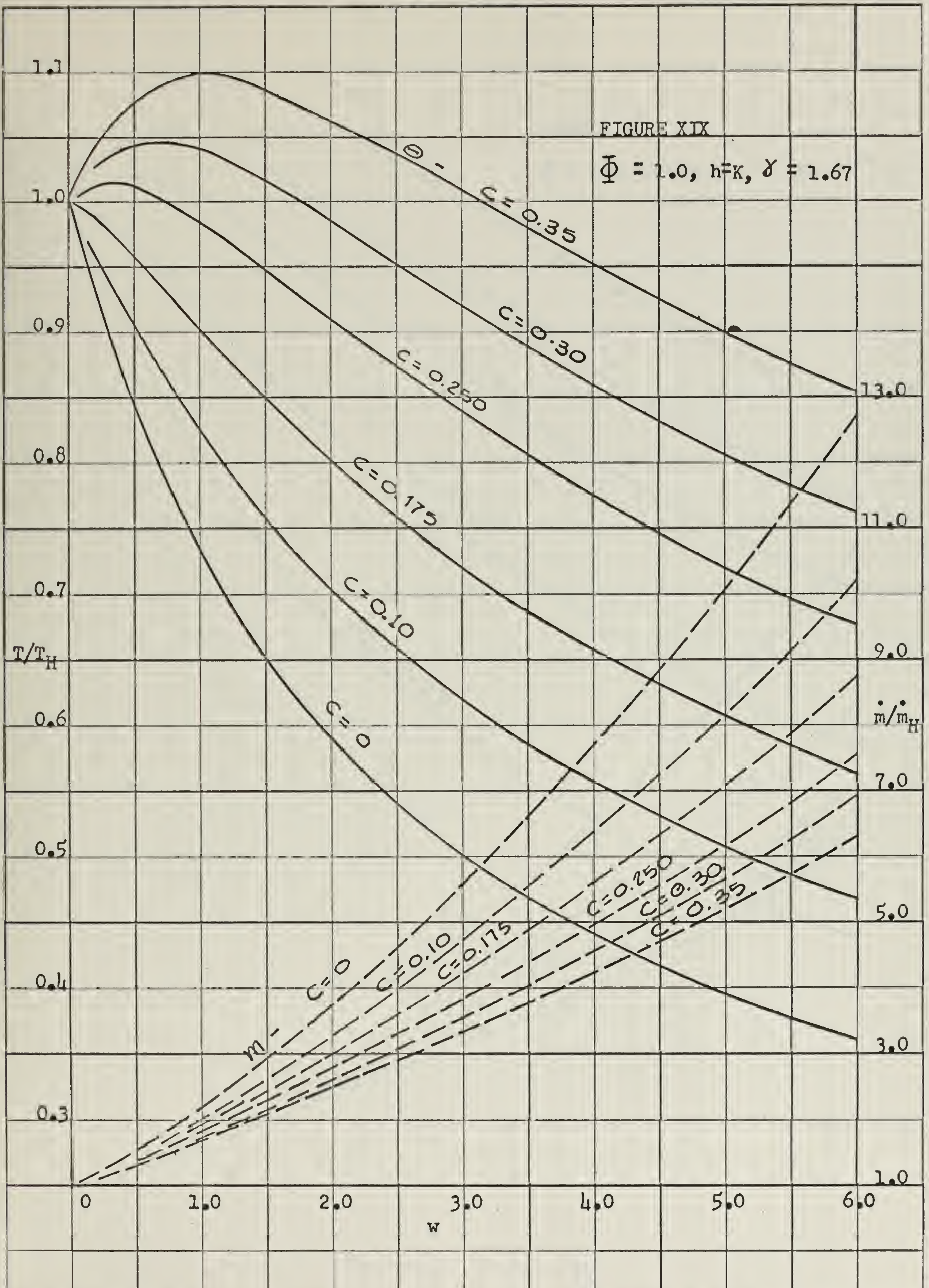


FIGURE XIX
Φ = 1.0, h=K, γ = 1.67
T/T$_H$
$\dot{m}/\dot{m}_H$
w
θ -
C = 0.35
C = 0.30
C = 0.250
C = 0.175
C = 0.10
C = 0
m -
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
13.0
11.0
9.0
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0

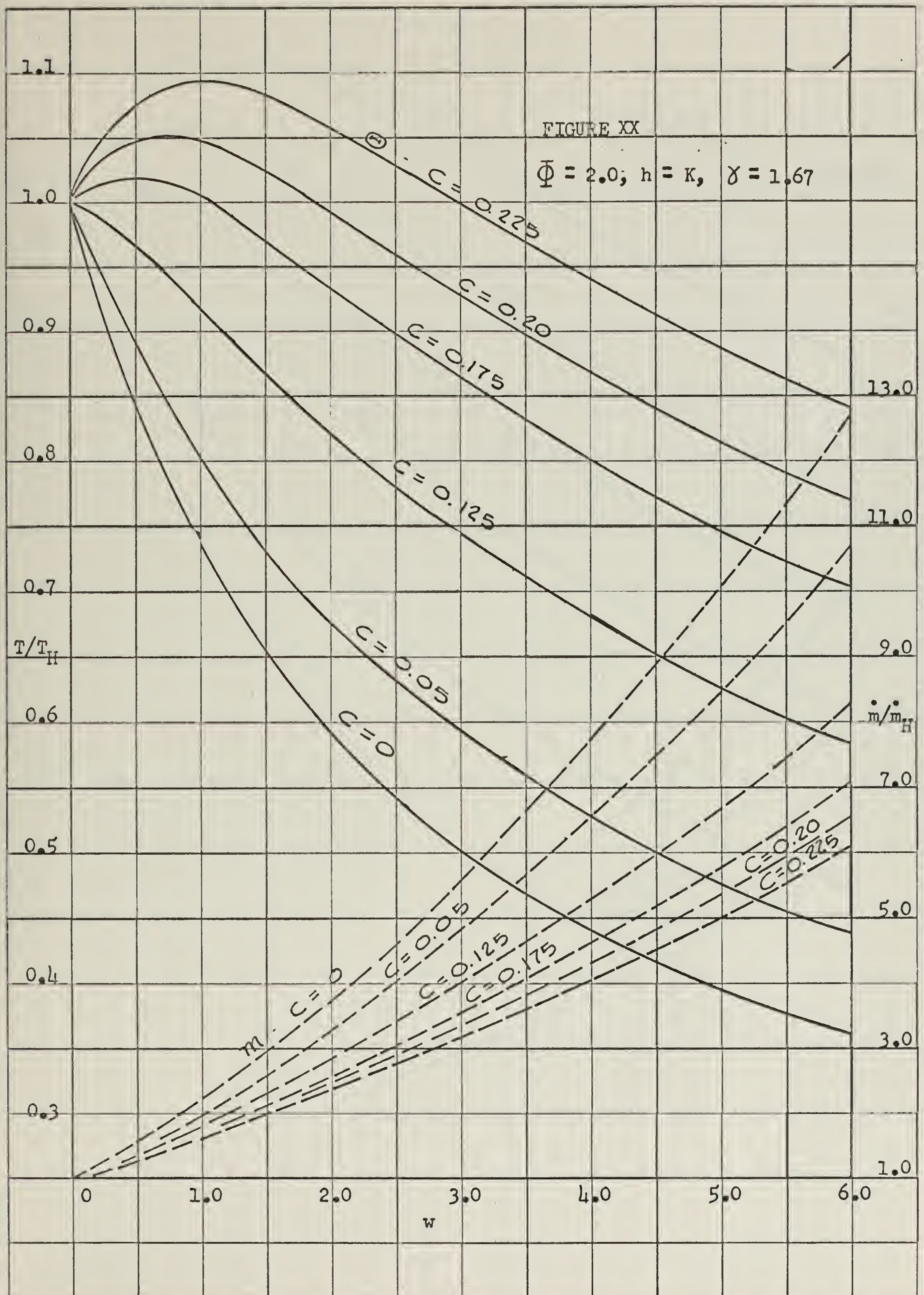


FIGURE XX
$\Phi = 2.0;\ h = K,\ \gamma = 1.67$
Θ . C = 0.225
C = 0.20
C = 0.175
C = 0.125
C = 0.05
C = 0
$\dot{m}$. C = 0
C = 0.05
C = 0.125
C = 0.175
C = 0.20
C = 0.225
T/T_H
$\dot{m}/\dot{m}_H$
w
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
13.0
11.0
9.0
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0

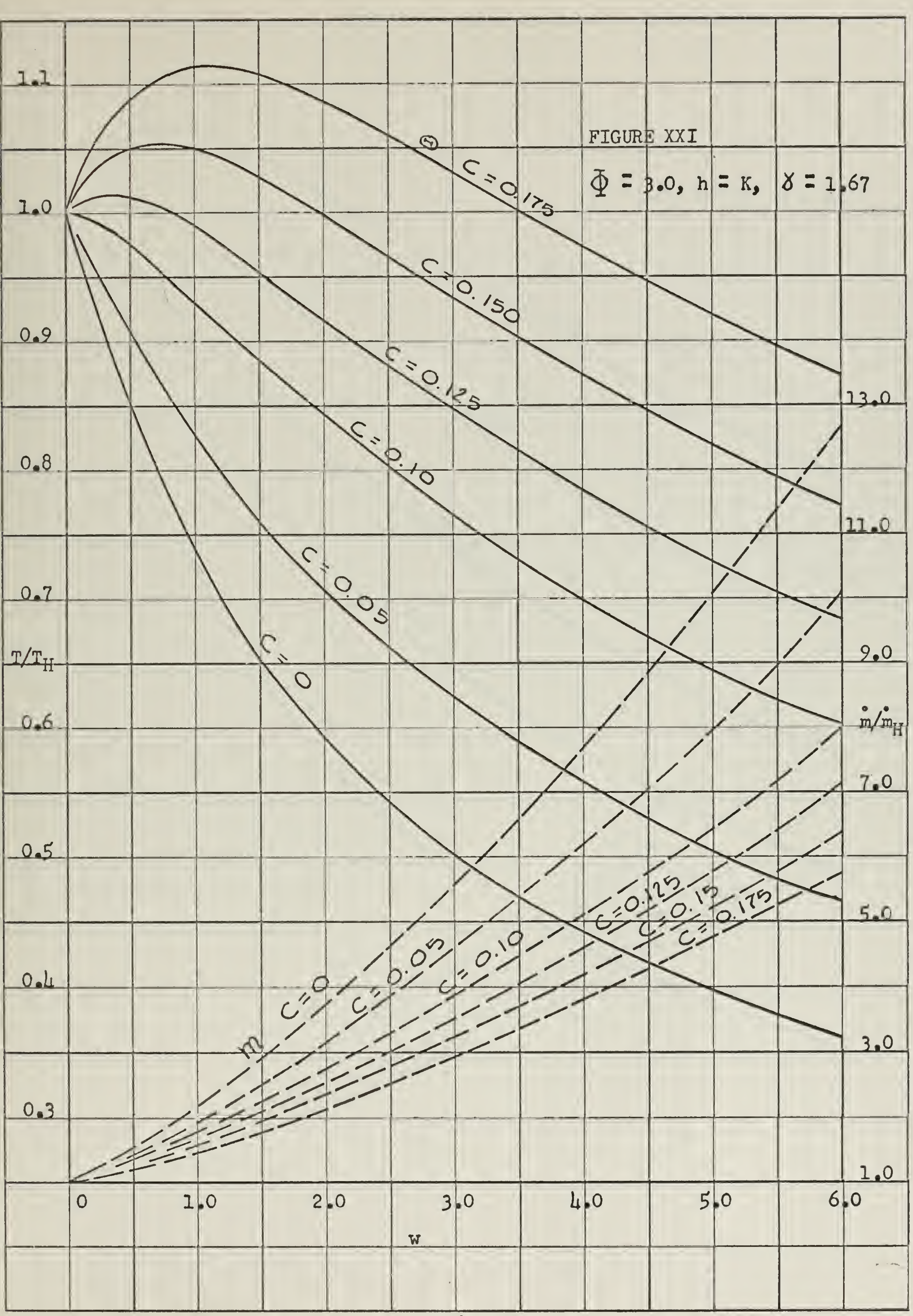


FIGURE XXI
Φ = 3.0, h = K, γ = 1.67
T/T_H
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
ṁ/ṁ_H
13.0
11.0
9.0
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0
w
Θ
C = 0.175
C = 0.150
C = 0.125
C = 0.10
C = 0.05
C = 0
ṁ
C = 0
C = 0.05
C = 0.10
C = 0.125
C = 0.15
C = 0.175

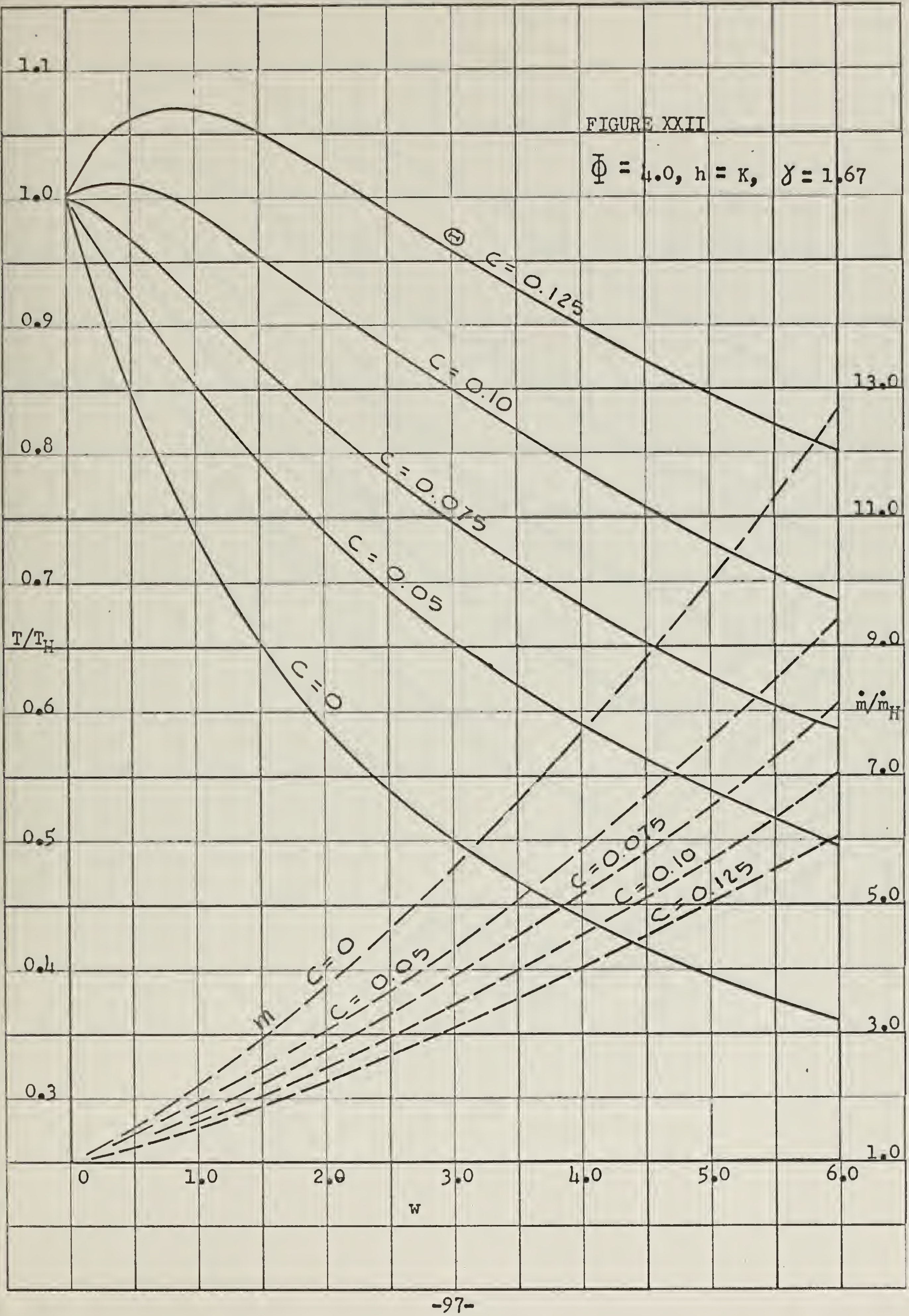


FIGURE XXII
$\Phi = 4.0$, $h = K$, $\gamma = 1.67$
Θ C = 0.125
C = 0.10
C = 0.075
C = 0.05
C = 0
$\dot{m}$ C = 0
C = 0.05
C = 0.075
C = 0.10
C = 0.125
T/T_H
$\dot{m}/\dot{m}_H$
w
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
13.0
11.0
9.0
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0

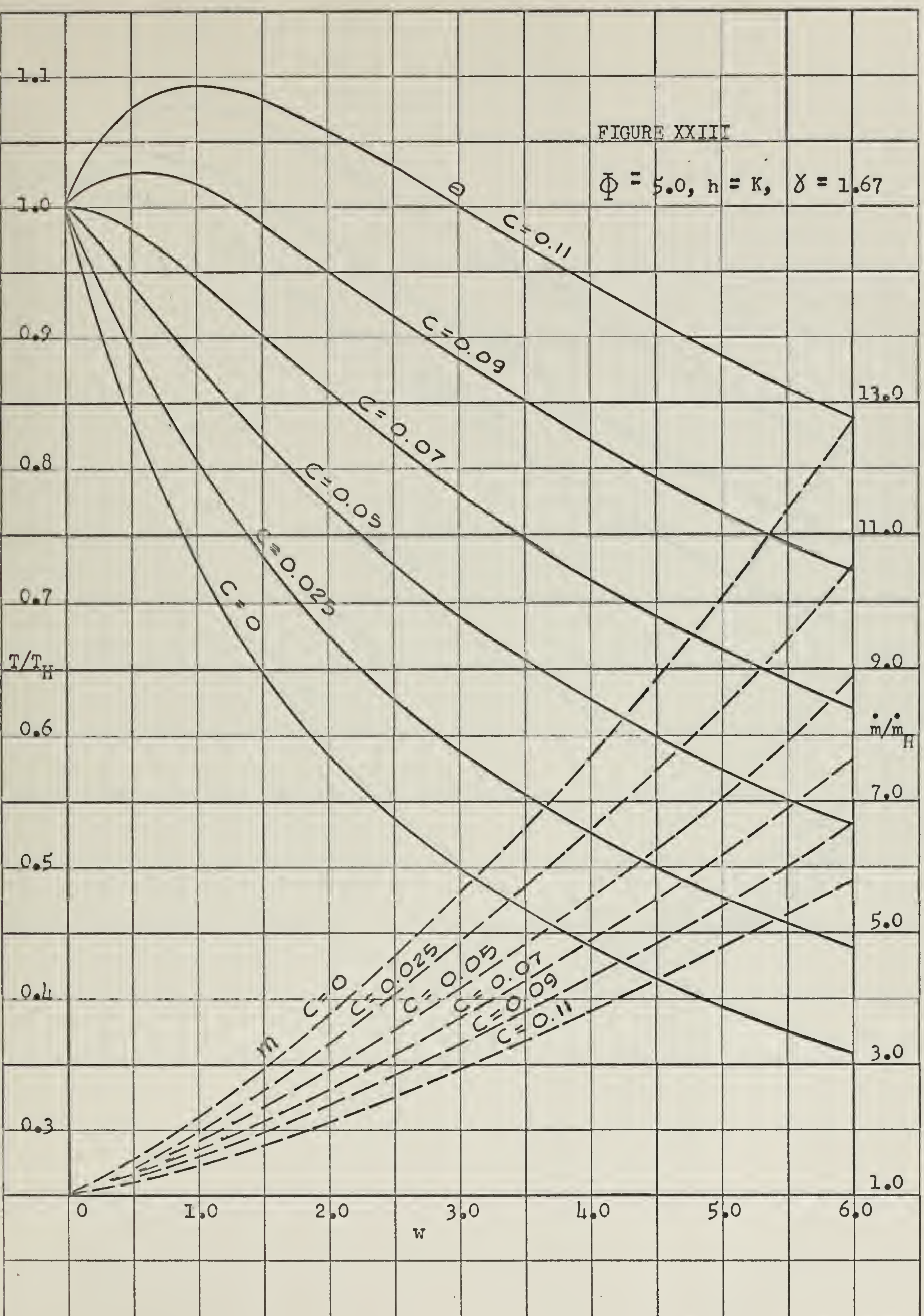


FIGURE XXIII
Φ = 5.0, h = K, γ = 1.67
1.1
1.0
0.9
0.8
0.7
T/T_H
0.6
0.5
0.4
0.3
13.0
11.0
9.0
ṁ/ṁ_H
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0
w
C=0.11
C=0.09
C=0.07
C=0.05
C=0.025
C=0
m
C=0
C=0.025
C=0.05
C=0.07
C=0.09
C=0.11

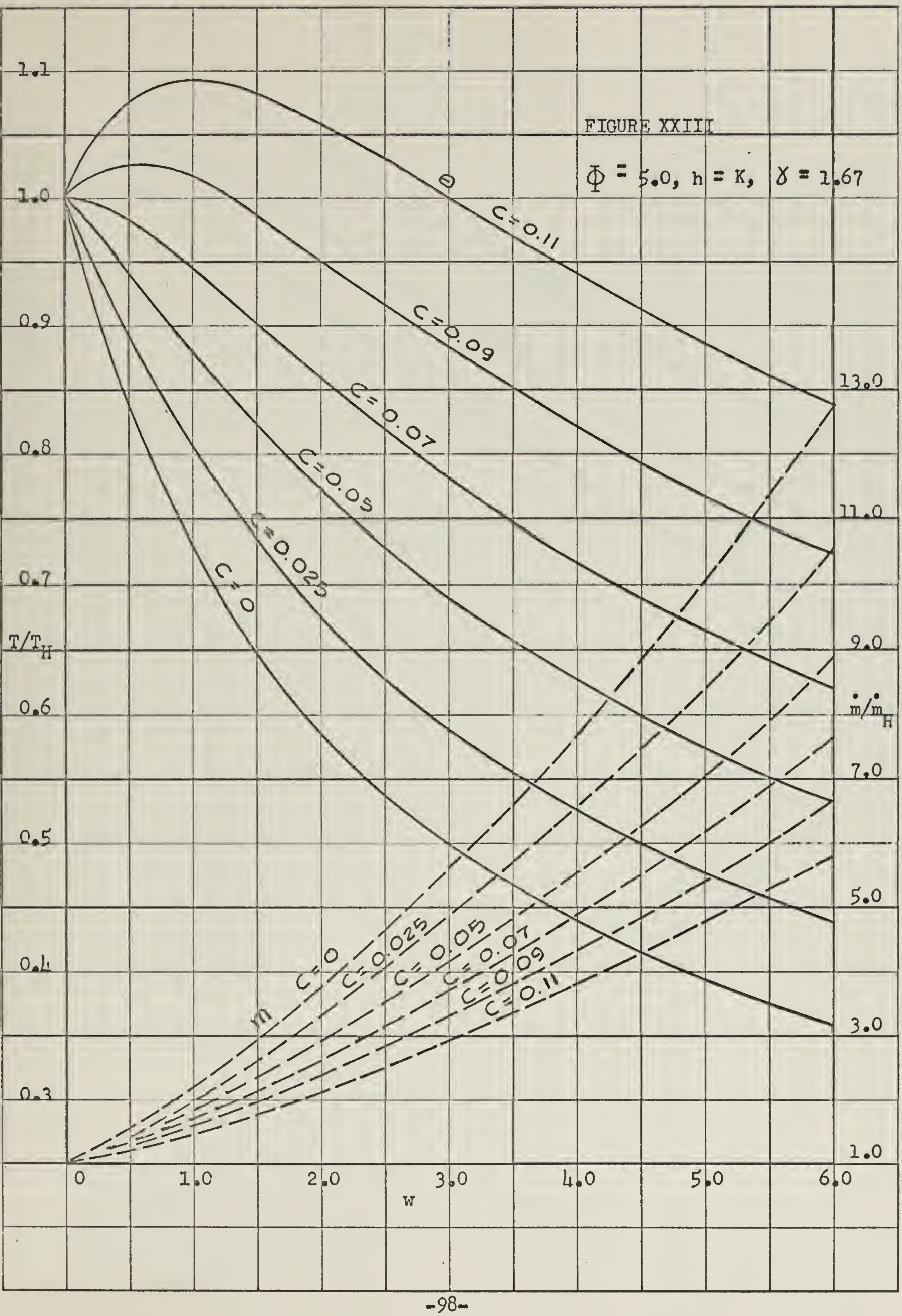


FIGURE XXIII
Φ = 5.0, h = K, γ = 1.67
1.1
1.0
0.9
0.8
0.7
T/T_H
0.6
0.5
0.4
0.3
0
1.0
2.0
3.0
4.0
5.0
6.0
w
13.0
11.0
9.0
ṁ/ṁ_H
7.0
5.0
3.0
1.0
θ
C=0.11
C=0.09
C=0.07
C=0.05
C=0.025
C=0
m
C=0
C=0.025
C=0.05
C=0.07
C=0.09
C=0.11

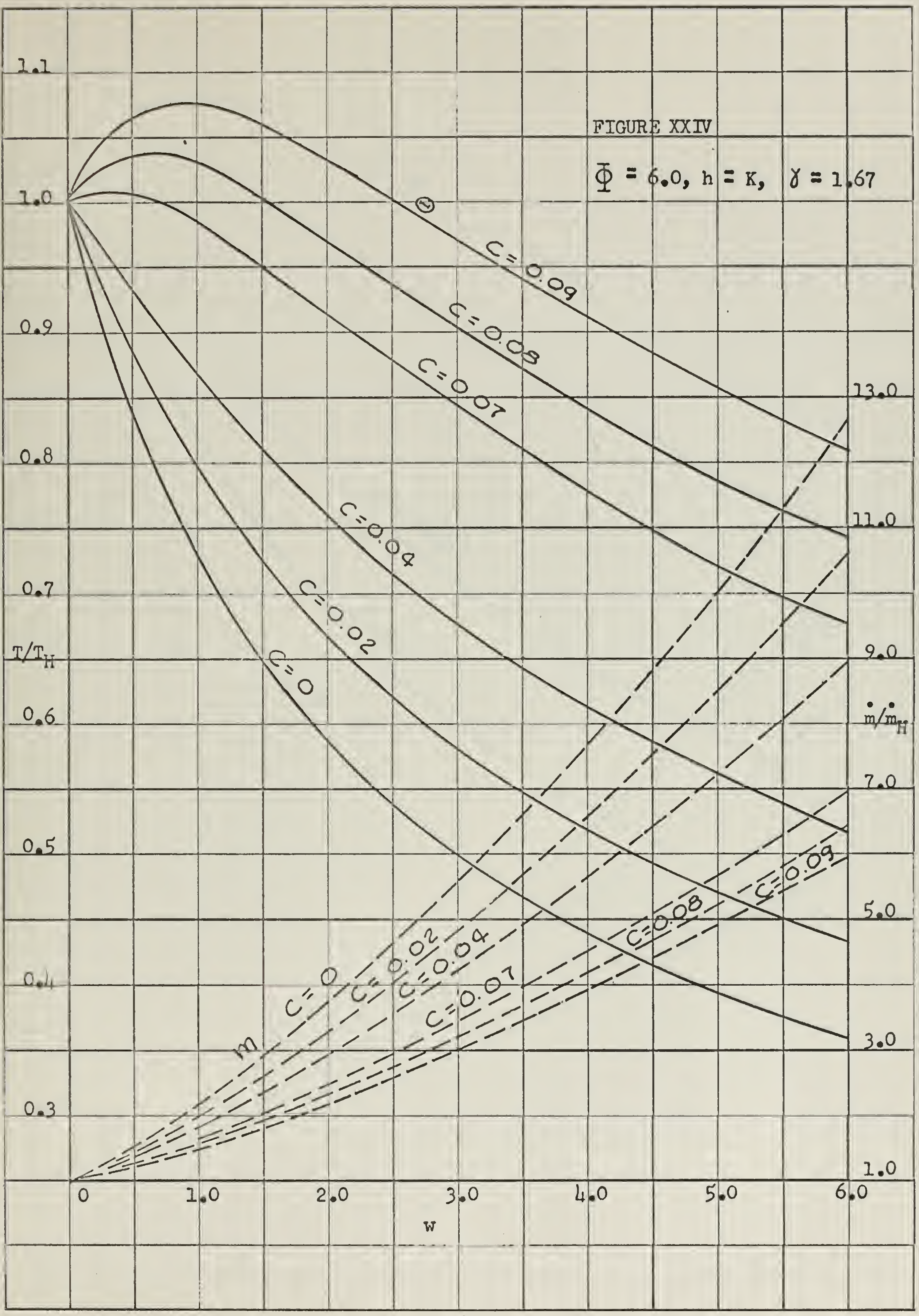


FIGURE XXIV
$\Phi = 6.0$, $h = K$, $\gamma = 1.67$
T/T_H
$\dot{m}/\dot{m}_H$
w
C=0.09
C=0.08
C=0.07
C=0.04
C=0.02
C=0
m
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
13.0
11.0
9.0
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0

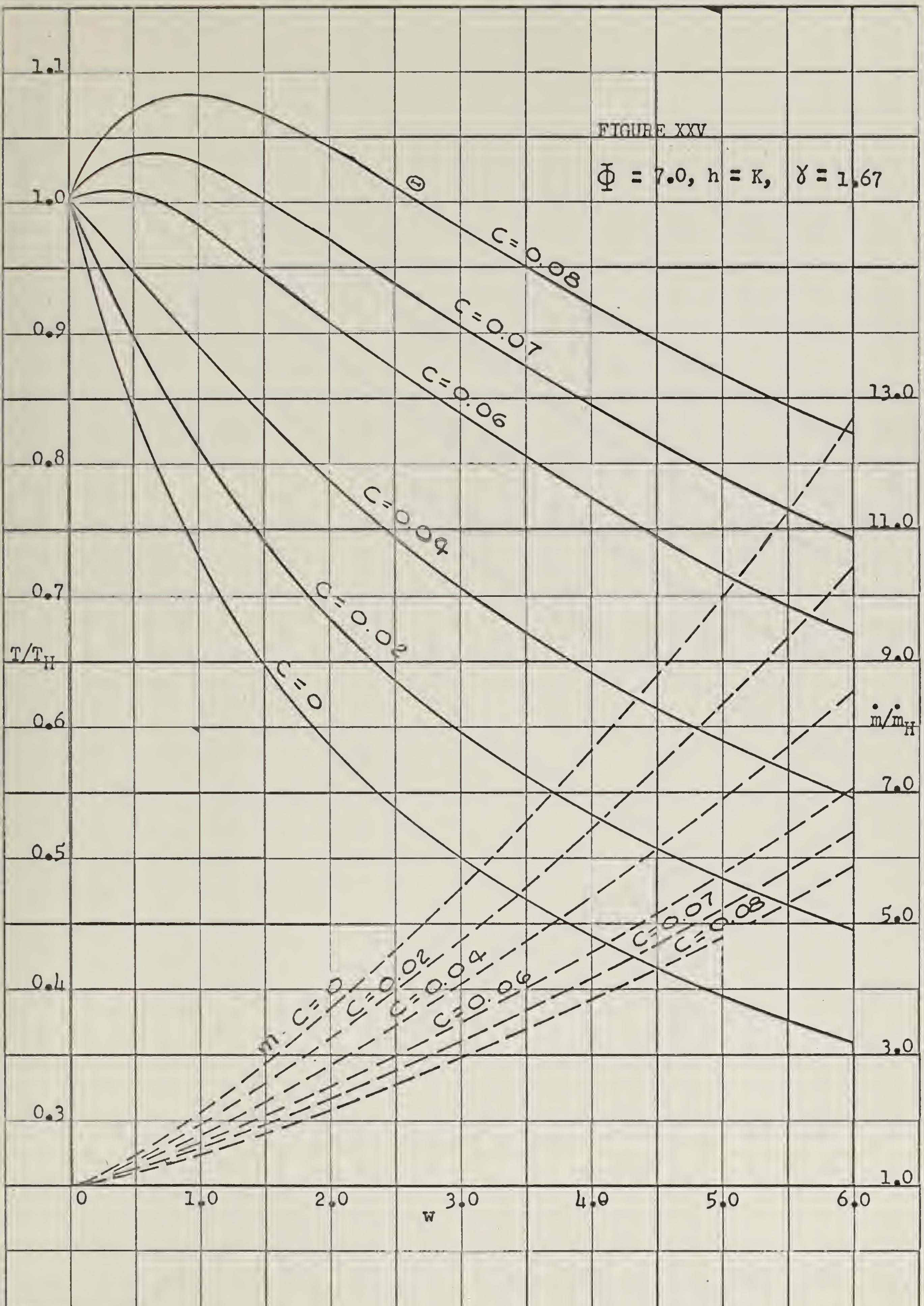


FIGURE XXV
$\Phi = 7.0,\ h = K,\ \gamma = 1.67$
Θ
C=0.08
C=0.07
C=0.06
C=0.04
C=0.02
C=0
m. C=0
C=0.02
C=0.04
C=0.06
C=0.07
C=0.08
T/T_H
$\dot{m}/\dot{m}_H$
w
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
13.0
11.0
9.0
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0

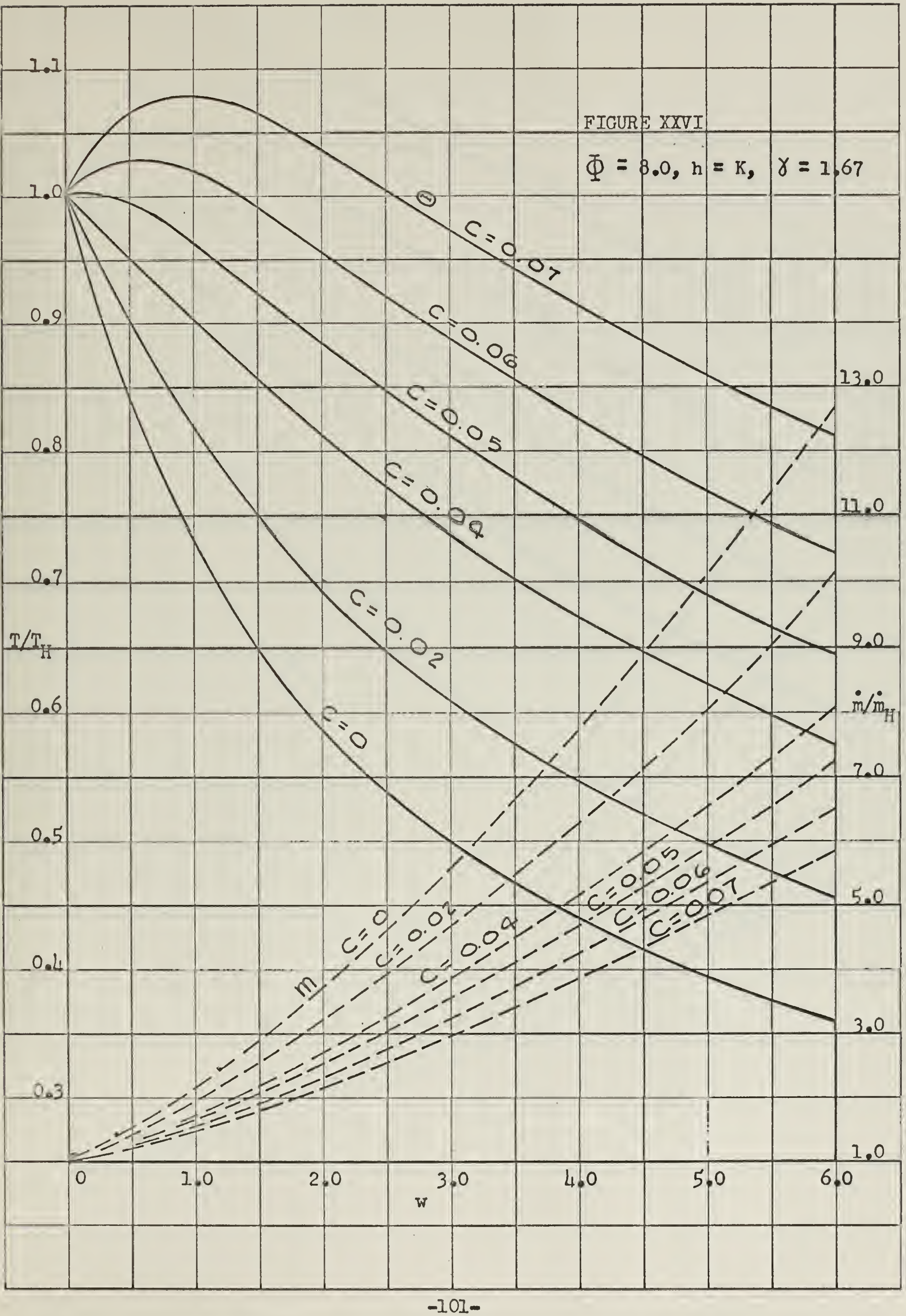


FIGURE XXVI
Φ = 8.0, h = K, γ = 1.67
1.1
1.0
0.9
0.8
0.7
T/T_H
0.6
0.5
0.4
0.3
13.0
11.0
9.0
ṁ/ṁ_H
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0
w
Θ
C=0.07
C=0.06
C=0.05
C=0.04
C=0.02
C=0
m
C=0
C=0.02
C=0.04
C=0.05
C=0.06
C=0.07

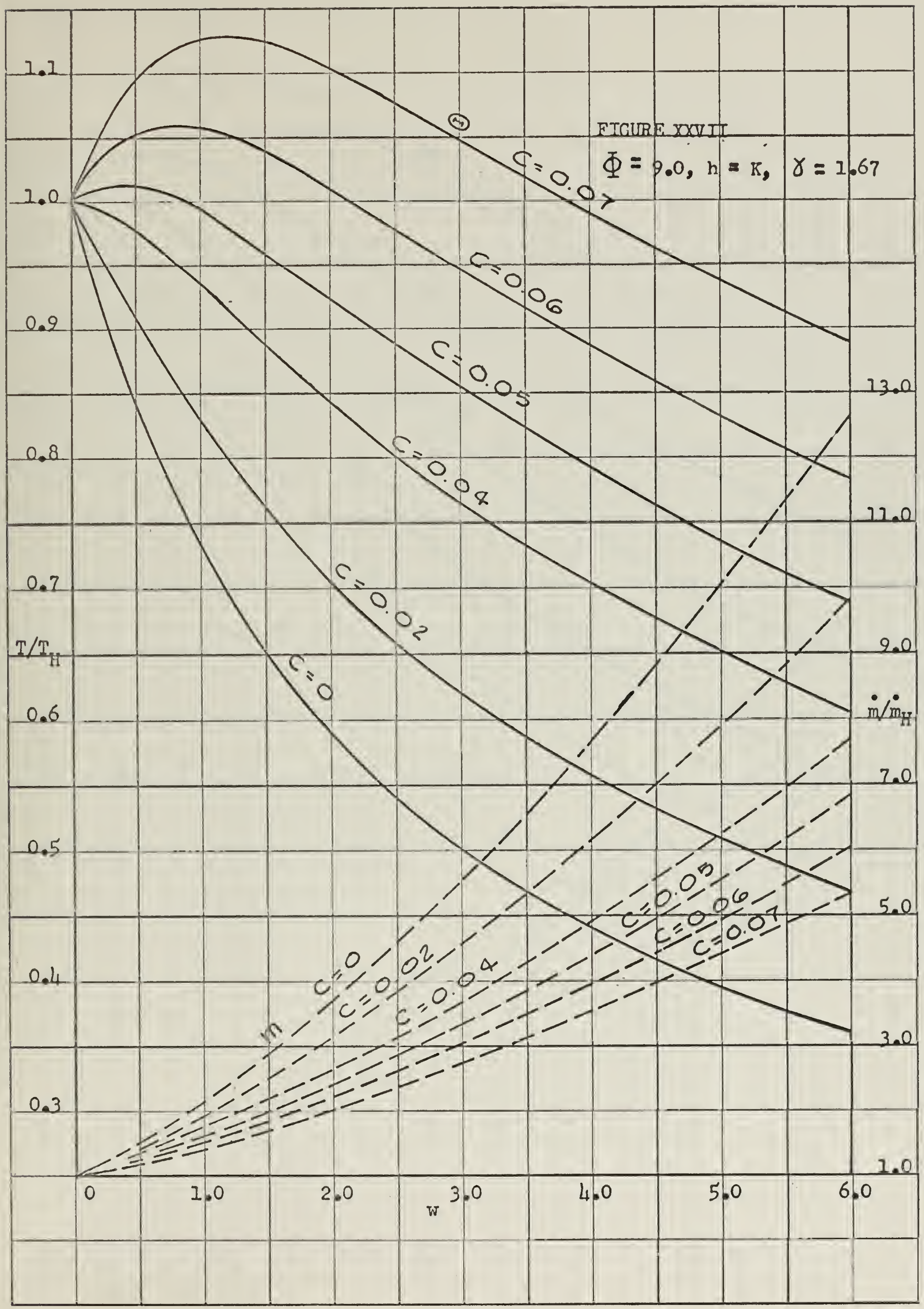


FIGURE XXVII
$\Phi = 9.0$, $h = K$, $\gamma = 1.67$
T/T_H
$\dot{m}/\dot{m}_H$
w
C=0.07
C=0.06
C=0.05
C=0.04
C=0.02
C=0
m
C=0
C=0.02
C=0.04
C=0.05
C=0.06
C=0.07
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
13.0
11.0
9.0
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0

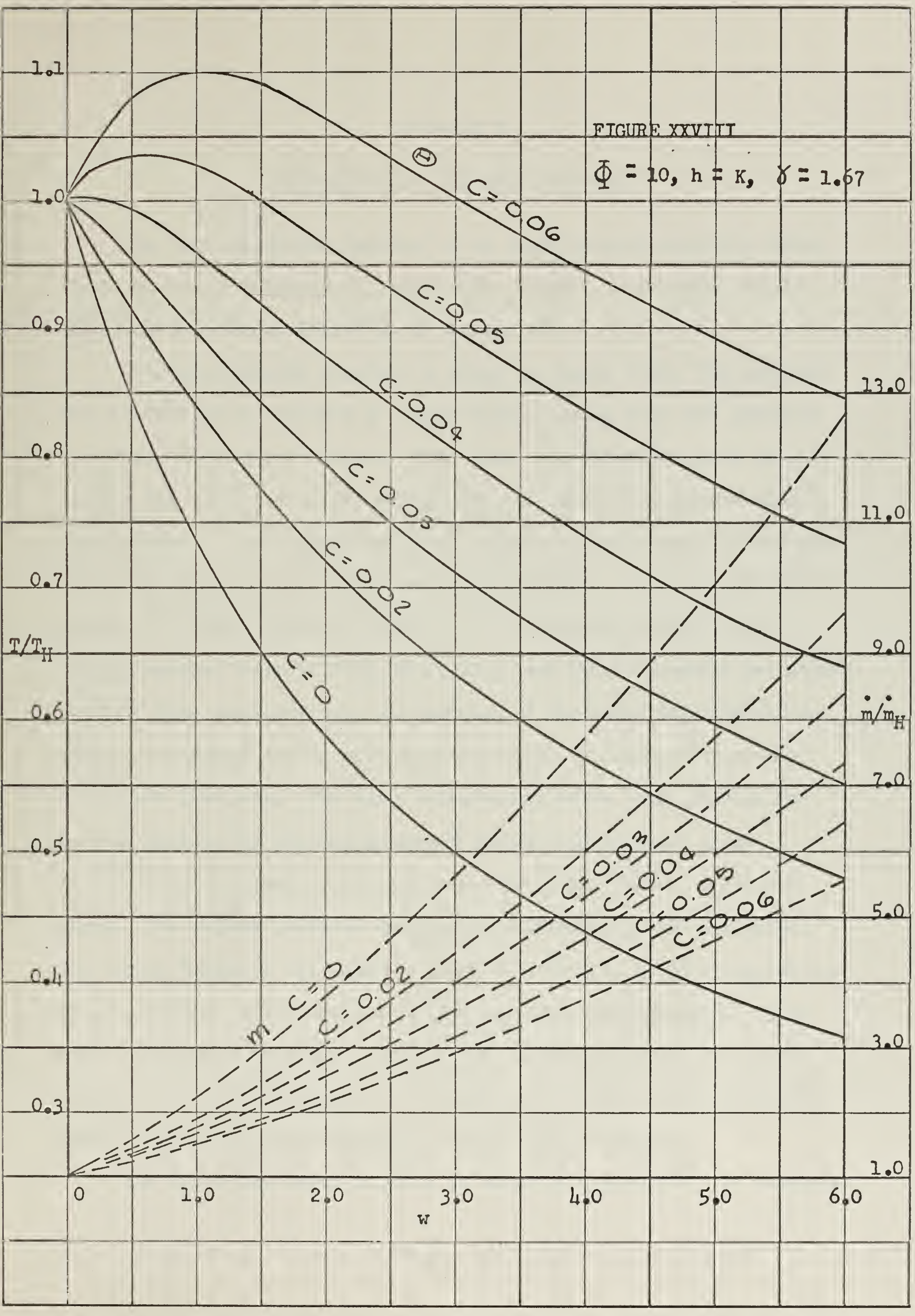


FIGURE XXVIII
Φ = 10, h = K, γ = 1.67
Θ
C=0.06
C=0.05
C=0.04
C=0.03
C=0.02
C=0
m C=0
C=0.02
C=0.03
C=0.04
C=0.05
C=0.06
1.1
1.0
0.9
0.8
0.7
T/T_H
0.6
0.5
0.4
0.3
13.0
11.0
9.0
m/m_H
7.0
5.0
3.0
1.0
0
1.0
2.0
3.0
4.0
5.0
6.0
w

APPENDIX D

DESCRIPTION OF THE TEST APPARATUS

The test apparatus consisted of an inlet/exhaust manifold, Pulse Tube, heater, heat exchanger, and vacuum chamber. A schematic of the apparatus is shown in Figure IV of Chapter IV.

The inlet/exhaust manifold is shown in Figure XXIX. The manifold was constructed of standard $\frac{1}{4}$" copper-tubing, using both soft soldered and flared fitting connections. Four Asco solenoid-valves were used to control the flow, two on the supply side and two on the exhaust side of the manifold. These valves had 3/32" orifice. The solenoid valves were cycled by a cam operated micro-switch arrangement. The cams were belt-driven by a small, electric motor through a four-step pulley which allowed cycling rates of 22.4, 44.7, 82.3, and 158.0 reversals per minute. Two 3/8" Hoke needle-valves, one upstream of the inlet solenoid-valves and one downstream of the exit solenoid-valves, provided fine control of the gas flow rate. The check valve/manual valve loops allowed for reading maximum and minimum pressures without any pressure gauge fluctuation. The maximum pressure was read on a 4"-Marsh, 0-300 psi gauge. The minimum pressure was read on a 3"-Marsh, 0-100 psi gauge. The copper-tubing of the manifold mated to a 1" O.D. by $2\frac{1}{2}$"-long, hollow, brass cylinder. The lower end of this cylinder was machined to allow an "O" ring to be inserted. This end of the cylinder was fitted into the open-end of the regenerator. Several short lengths of tubing were fitted inside the brass cylinder to act as flow smoothers.

The Pulse Tube drawn to scale is shown in Figure XXX. It consisted

FIGURE XXIX

INLET/EXHAUST MANIFOLD

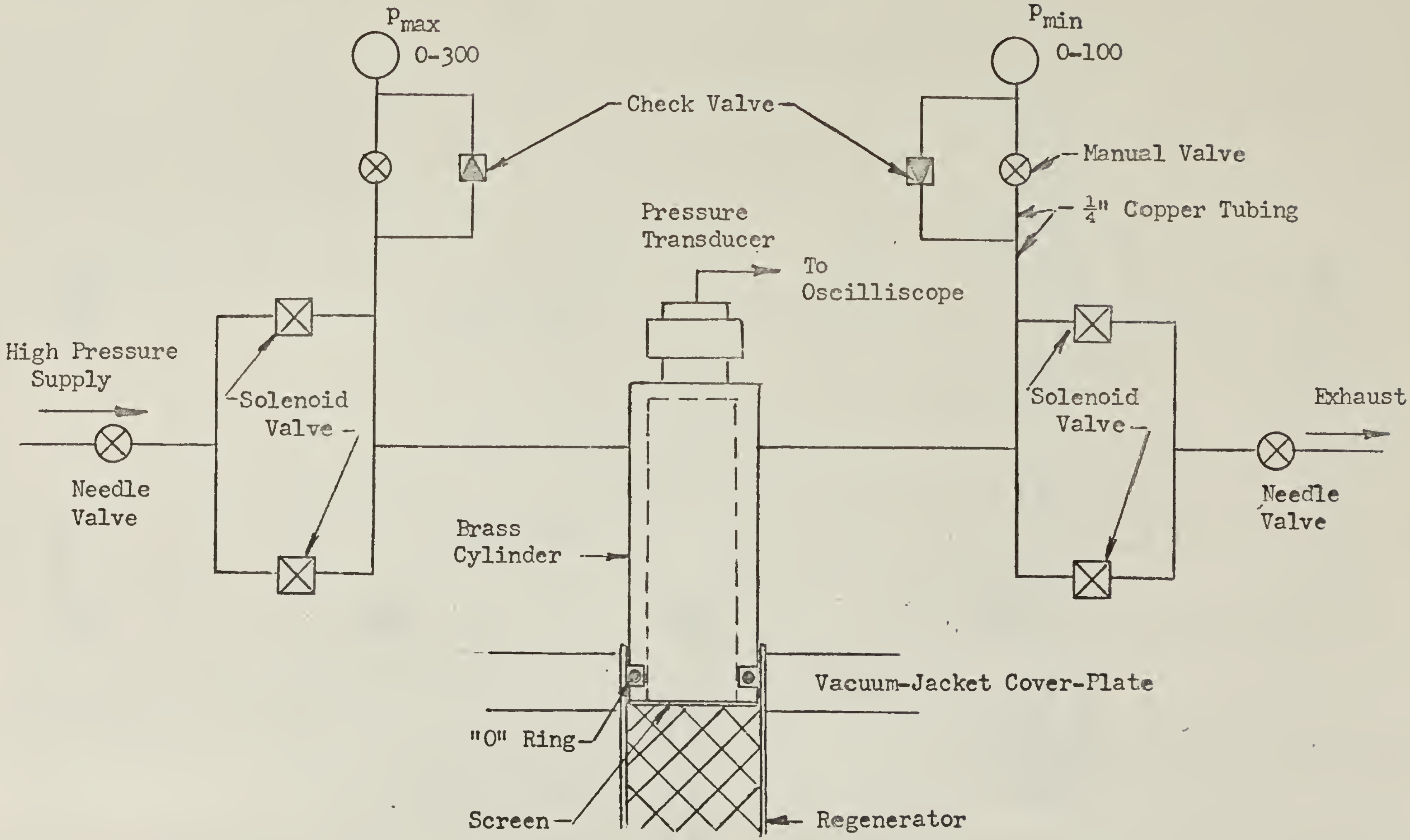
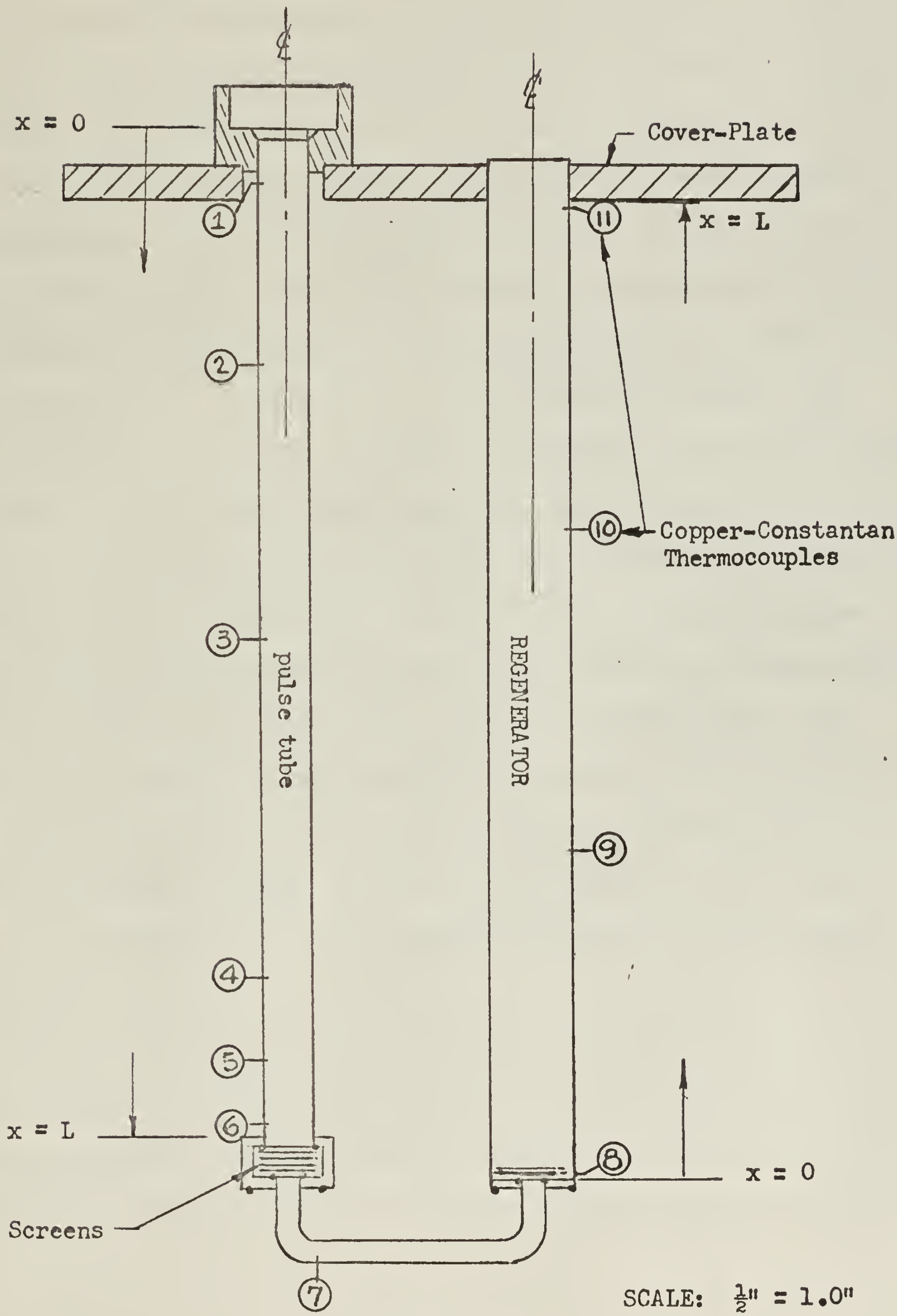


FIGURE XXX

SCALE DRAWING OF PULSE TUBE



of a regenerator, a void tube (pulse tube), and a short length of connecting tubing. The regenerator was constructed from a 12"-long, stainless-steel tube, 1" O.D. by 0.020"-wall-thickness packed with 0.050"-diameter lead shot. The pulse tube consisted of a 12"-long, stainless-steel tube, 5/8" O.D. by 0.016"-wall-thickness. The regenerator and pulse tube were connected at the bottom by a $\frac{1}{4}$" O.D. by 0.020"-wall-thickness, stainless-steel tube. On the pulse tube end of the connecting tubing was a 1" I.D. by 1"-long, stainless-steel cylinder, which was filled with fine-mesh, bronze screens to serve as a flow smoothing device. Several layers of this screening were also placed at the bottom of the regenerator for the same purpose. The stainless-steel components were heli-arc welded as shown in Figure XXX. The top ends of the regenerator and pulse tube were soft soldered 3" apart to a brass, vacuum-jacket, cover-plate, which was 9" O.D. by 3/8"-thick. Eleven copper-constantan thermocouples were installed along the regenerator and pulse tube at locations indicated in Figure XXX. All the thermocouples were led out of the vacuum chamber through two 1"-diameter Latronic ceramic seals, which were soldered into pre-drilled holes in the cover plate. The thermocouple outputs were recorded by a Leeds and Northrup multipoint recorder. The pressure-time history within the Pulse Tube was measured by diaphragm-type pressure transducers mounted at the top of the regenerator and pulse tube. An oscilloscope was used to indicate the output of the pressure transducers.

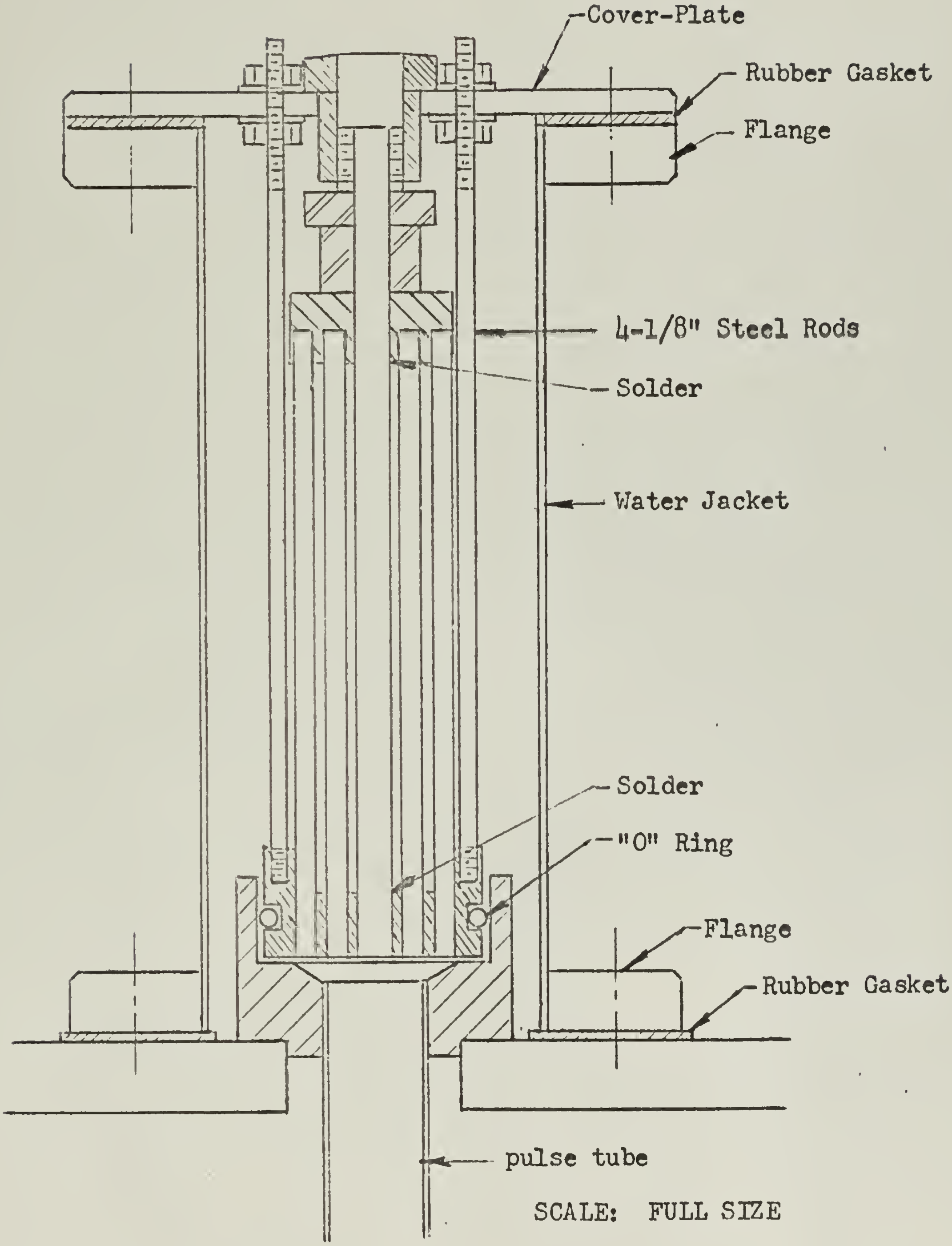
The heater was constructed as follows: the $\frac{1}{4}$" stainless-steel tubing connecting the regenerator and pulse tube was coated with a thin layer of "Sauereisen Insa-Lute Adhesive Cement", a liquid-porcelain type of material. After the cement had hardened, approximately two meters of

a high resistance wire were wrapped on top of it. The heater wire was connected to two 32-gauge, copper, lead-in wires. Then the entire heater assembly was coated again with a second layer of cement. The lead-in wires were lead through a Latronic ceramic seal installed in the vacuum-jacket cover plate. The total resistance of the heater circuit was 121 ohms. The heater circuit external to the vacuum jacket consisted of a 0-140 volt variac for fine voltage control. A milliampmeter, voltmeter, and wattmeter were installed in the external circuit for power measurements.

The heat exchanger at the top end of the pulse tube was constructed as shown in Figure XXXI. Thin-wall, copper-nickel tubing 1/8" O.D. was used to promote good heat transfer. The working gas was funneled into the tubes, which were approximately $3\frac{1}{2}$" in length and which were dead-ended in a brass tube sheet approximately 1" in diameter by $\frac{1}{2}$"-thick. The entire tube assembly was surrounded by a water jacket, which was attached to the cover plate. Cooling water was pumped through the water jacket at the rate of 0.5 gallons per minute.

FIGURE XXXI

HEAT EXCHANGER



SCALE: FULL SIZE